This exam has 9 questions for a total of 100 points. Show all you your work! In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number.

You may not use a calculator on this exam. Please turn off and put away your cell phone.
1. (11 points) Solve the following initial value problem. Give your answer in the explicit form.

\[ y' = \frac{te^{-t} + 2 \cos(t)}{2y - 8}, \quad y(0) = 3. \]
2. (10 points) Consider the differential equation
\[ y^4 - 3x^2 \tan(y) + (4xy^3 - x^3 \sec^2(y) + 2e^{2y})y' = 0. \]

(a) (3 points) Verify that it is an exact equation.

(b) (7 points) Find the solution of this equation satisfying \( y(-2) = 0 \). You may leave your answer in an implicit form.
3. (12 points) A 20 kg cannonball is shot straight up with an initial velocity of 60 m/s off the very top of a castle’s tower. Suppose the force of air resistance is given by $2|v|$, and $g = 10$ is the gravitational constant. Let the downward direction be the positive direction.

(a) (4 points) Set up an initial value problem that models the velocity of the cannonball as a function of time (until it lands).

(b) (4 points) Solve the initial value problem.

(c) (2 points) Given unlimited time and distance to travel, as $t \to \infty$, what would the cannonball’s terminal velocity be?

(d) (2 points) Suppose the castle’s tower is 50 meters tall, find the displacement of the cannonball as a function of time (until it lands).
4. (12 points) Suppose a 120-gallon tank initially contains 90 gal of water with 90 ounces of salt dissolved in it. Salt water (with a concentration of 2 oz/gal) enters the tank at a rate of 4 gal/min. The well-mixed solution flows out of the tank at a rate of 3 gal/min.

(a) (4 points) Set up an initial value problem that models the amount of dissolved salt in the tank at any time, until the tank is full.

(b) (4 points) Solve the initial value problem.

(c) (2 points) What is the concentration of the dissolved salt in the tank at any time before the tank is full?

(d) (2 points) How much salt is in the tank when it is full?
5. (12 points) Consider the autonomous differential equation
\[ y' = (y + 3)^2(1 - y^2). \]

(a) (6 points) Find all of its equilibrium solutions. Classify the stability of each equilibrium solution. Justify your answer.

(b) (2 points) If \( y(-3) = -1 \), then what is \( \lim_{t \to \infty} y(t) \)?

(c) (2 points) If \( y(1) = -2 \), what is \( \lim_{t \to \infty} y(t) \)?

(d) (2 points) If \( y(100) = \alpha \) and \( \lim_{t \to \infty} y(t) = 1 \), what are all the possible values of \( \alpha \)?
6. (9 points) Consider the second order linear equation

\[ y'' - 20y' + 100y = 0. \]

(a) (3 points) Find the general solution of the equation.

(b) (4 points) Find the solution satisfying the initial conditions \( y(570) = 2, \ y'(570) = 19. \)

(c) (2 points) For your answer from (b), determine \( \lim_{t \to \infty} y(t). \)
7. (10 points) Given that \( y_1(t) = t^3 \cos(\ln t) \) and \( y_2(t) = t^3 \sin(\ln t) \) are both solutions of the second order homogeneous linear equation
\[
t^2 y'' - 5t y' + 10y = 0, \quad t > 0.
\]
Determine whether each of the following statements is true or false. State a brief reason that justifies each answer.
(a) (2 points) Their Wronskian \( W(y_1, y_2)(t) = 0. \)
(b) (2 points) \( y_1 \) and \( y_2 \) form a set of fundamental solutions of this equation.
(c) (2 points) \( y_3(t) = t^6 \cos(\ln t) \sin(\ln t) \) is also a solution of the equation.
(d) (2 points) \( y_4(t) = 0 \) is also a solution of the equation.
(e) (2 points) Given any pair of initial conditions \( y(1) = \alpha \) and \( y'(1) = \beta \) there is always a unique linear combination of \( y_1 \) and \( y_2 \) that satisfies the given initial value problem at any point on the interval \( (0, \infty) \).
8. (14 points) Consider the second order nonhomogeneous linear equation

\[ y'' - 4y' + 5y = 3t + e^{2t}. \]

(a) (3 points) Find \( y_c(t) \), the solution of its corresponding homogeneous equation.

(b) (7 points) Find its general solution by using the Method of Undetermined Coefficients.

(c) (4 points) What is the form of particular solution \( Y \) that you would use to solve the following equation using the Method of Undetermined Coefficients? \textbf{DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.}

\[ y'' - 4y' + 5y = t^2 e^t \cos 2t - 2te^{2t} \sin t. \]
9. (10 points) Consider a mass-spring system described by the equation

\[ 5u'' + \gamma u' + ku = 0, \quad \gamma \geq 0, \quad k > 0. \]

Answer the following questions. Be sure to justify your answer. Full credit will not be given without supporting work.

(a) (2 points) Suppose the spring was stretched 2 meters by the mass to its equilibrium position. Find the value of \( k \). You may use \( g = 10 \) as the gravitational constant.

(b) (2 points) Suppose \( k = 20 \). For what value(s) of \( \gamma \) would the system be critically damped?

(c) (2 points) Suppose \( \gamma = 15 \) and \( k = 10 \). Will any nonzero solution of the equation cross the equilibrium position more than once?

(d) (2 points) Suppose \( \gamma = 10 \) and \( k = 50 \). Find the quasi period of the system.

(e) (2 points) Consider the displacement of the system in part (d) above, determine \( \lim_{t \to \infty} u(t) \).