

MATH 251  
Examination I  
July 8, 2008

Name: \_\_\_\_\_  
Student Number: \_\_\_\_\_  
Section: \_\_\_\_\_

This exam has 11 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.** The point value for each question is in parentheses to the right of the question number.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

**Do not write in this box.**

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Total: _____

1. (5 points) Which of the following first order differential equation is **both** linear and autonomous?

(a)  $y' + ty = 0$

(b)  $y' - y^3 = 2$

(c)  $y' + 4y = \pi$

(d)  $y' + e^y - 5t = 0$

2. (5 points) Consider the initial value problem

$$(t^2 - 16)y' + \sin\left(\frac{t}{5}\right)y = \frac{t+1}{t-1}, \quad y(\pi) = \frac{1}{2}.$$

Without solving the equation, what is the largest interval in which a unique solution is guaranteed to exist?

(a)  $(1, \infty)$

(b)  $(-4, 4)$

(c)  $(1, 4)$

(d)  $(-\infty, 1)$

3. (5 points) Which of the following pairs of functions is **not** linearly independent on  $(-\infty, \infty)$ ?

(a)  $e^{\frac{1}{3}t}$ ,  $e^{-\frac{1}{3}t}$

(b)  $e^{-t}$ ,  $2e^{-(t-4)}$

(c)  $3\cos(\pi t)$ ,  $2\sin(\pi t)$

(d)  $5$ ,  $e^{5t}$

4. (5 points) What is the general solution of

$$y'' - 6y' + 10y = 0?$$

(a)  $C_1 e^{-t} + C_2 e^{-5t}$

(b)  $C_1 e^{3t} + C_2 t e^{3t}$

(c)  $C_1 e^{-t} \cos 3t + C_2 e^{-t} \sin 3t$

(d)  $C_1 e^{3t} \cos t + C_2 e^{3t} \sin t$

5. (10 points) Find, **in explicit form**, the solution of the initial value problem

$$y' = \frac{3x^2 - \sin x}{y + 2}, \quad y(0) = -4.$$

6. (12 points) Iodine solution is being prepared in a mixing vat. The vat is initially filled with 80 liters of pure ethyl alcohol (that is, ethanol). Additional ethyl alcohol containing 20 grams per liter of iodine flows into the vat at a rate of 1 liter per minute. The well-stirred iodine solution flows out of the vat at a rate of 2 liters per minute.
- (a) (4 points) Let  $Q(t)$  denote the amount of dissolved iodine in the vat at any time  $t$ ,  $0 < t < 80$ . Write down an initial value problem (be sure to give both a differential equation and an initial condition) that  $Q(t)$  must satisfy.
- (b) (8 points) Solve the initial value problem to find  $Q(t)$ .

7. (12 points) Consider the autonomous differential equation

$$y' = (2 + y)(2y - 9)(y - 8).$$

(a) (3 points) Find all equilibrium solutions.

(b) (5 points) Classify the stability of each equilibrium solution. Justify your answer.

(c) (2 points) If  $y(2\pi) = 0$ , what is  $\lim_{t \rightarrow \infty} y(t)$ ?

(d) (2 points) If  $y(-20) = 8$ , what is  $\lim_{t \rightarrow \infty} y(t)$ ?

8. (12 points)

(a) (4 points) Consider the differential equation

$$(\pi y \cos(\pi x) + 3x^2 y - 2e^x) + (\sin(\pi x) + x^3 + 5) \frac{dy}{dx} = 0.$$

Verify that this equation is an exact equation.

(b) (8 points) Find the solution of the equation above that also satisfies the initial condition  $y(2) = -1$ . You may leave your answer in implicit form.

9. (10 points) Consider the initial value problem

$$y'' - 8y' + 16y = 0, \quad y(0) = 3, \quad y'(0) = 10.$$

(a) (8 points) Find the solution,  $y(t)$ , of this initial value problem.

(b) (2 points) What is  $\lim_{t \rightarrow \infty} y(t)$ ?

10. (10 points) Given that  $y_1(t) = t^2$  is a known solution of the second order linear differential equation

$$t^2 y'' - 3ty' + 4y = 0, \quad t > 0.$$

Find the general solution of the equation.

11. (14 points) Consider the nonhomogeneous second order linear equation of the form

$$y'' - y' - 2y = g(t).$$

(a) (3 points) Find its complementary solution,  $y_c(t)$ .

(b) (7 points) Find a particular solution  $Y(t)$  that satisfies

$$y'' - y' - 2y = 3 \sin 2t.$$

(c) (4 points) Write down the correct choice of the **form** of particular solution that you would use to solve the equation below using the Method of Undetermined Coefficients. **DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.**

$$y'' - y' - 2y = t^3 e^{-t} - 7e^{2t} \cos 6t$$