

MATH 251
Midterm Exam I
June 29, 2006

Name: _____
Student Number: _____
Instructor: _____
Section: _____

This exam has 11 questions for a total of 100 points. There are **6** partial credit questions. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.**

THE USE OF CALCULATORS IS NOT PERMITTED IN THIS EXAMINATION.
At the end of the examination, the booklet will be collected.

Do not write in this box.

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Total: _____

1. (6 points) For each of the differential equations below state its order and whether it is linear or nonlinear.

(a) $y''' + t^3y' = e^t y$

(b) $y' = y(y + 1)$

(c) $y' - 2y''y = t - 1$

2. (5 points) The solution of the initial value problem

$$y' = -\frac{x}{y}, \quad y(0) = -2,$$

is

(a) $y = -\sqrt{x^2 + 4}$

(b) $y = \sqrt{x^2 + 4}$

(c) $y = -\sqrt{4 - x^2}$

(d) $y = \sqrt{4 - x^2}$

3. (5 points) All of the following pairs of functions are linearly independent on $(-\infty, \infty)$ EXCEPT

(a) e^{-t}, e^{4t}

(b) $e^t, e^t - 5$

(c) e^{3t}, e^{3t+3}

(d) $\cos(2t), \sin(2t + 4\pi)$

4. (5 points) The general solution of

$$y'' - 8y' + 16y = 0$$

is

(a) $y = C_1 e^{4t}$

(b) $y = C_1 e^{4t} + C_2 t e^{4t}$

(c) $y = C_1 e^{4t} + C_2 e^{-4t}$

(d) $y = C_1 e^{2t} + C_2 e^{8t}$

5. (5 points) Let $y(t)$ be the solution of the initial value problem

$$y'' - 6y' - 7y = 0, \quad y(0) = 5, \quad y'(0) = b.$$

Suppose $\lim_{t \rightarrow \infty} y(t) = 0$, find the value of b .

- (a) -5
- (b) 0
- (c) 10
- (d) 30

6. (12 points)

(a) (9 points) Solve the following initial value problem:

$$ty' - 2y = t^3 \cos t, \quad y(\pi) = 2.$$

(b) (3 points) According to the Existence and Uniqueness Theorem, what is the largest interval on which the solution in (a) is guaranteed to uniquely exist?

7. (12 points) Consider the first order autonomous equation:

$$\frac{dy}{dt} = (y - 6)^2 (y^2 - 25).$$

(a) (3 points) Find all equilibrium solutions.

(b) (5 points) For each equilibrium solution classify its stability. Justify your answer.

(c) (2 points) If $y(1) = 0$, what is $\lim_{t \rightarrow \infty} y(t)$?

(d) (2 points) Suppose $y(0) = y_0$, and $\lim_{t \rightarrow \infty} y(t) = 5$. Find the value(s) of y_0 .

8. (12 points) Consider the initial value problem:

$$6x^2 - 2xy + e^{x+y} + (e^{x+y} - x^2) \frac{dy}{dx} = 0, \quad y(1) = -1.$$

(a) (4 points) Verify that the equation is exact.

(b) (8 points) Solve this initial value problem. You may leave your answer in implicit form.

9. (10 points) Solve the following initial value problem:

$$y'' + 4y' + 13y = 0, \quad y(0) = 3, \quad y'(0) = -9.$$

Be sure to express your solution in terms of real-valued functions only.

10. (12 points) Consider the second order linear equation

$$t^2 y'' - ty' + y = 0, \quad t > 0.$$

Given that $y_1(t) = t$ is a known solution, find the general solution of the equation.

11. (16 points) A swimming pool is initially filled with 400 m^3 of fresh water. At $t = 0$ water containing 50 g/m^3 of chlorine starts to flow into the swimming pool at a rate of 2 m^3 per minute. Well-mixed water is drained from the pool at the same rate.

(a) (4 points) Set up an initial value problem modeling this process.

(b) (8 points) Solve the initial value problem.

(c) (2 points) Find the time when the chlorine concentration within the pool reaches 25 g/m^3 .

(d) (2 points) (As $t \rightarrow \infty$) what is the limiting amount of chlorine that will be in the swimming pool?