This exam has 12 questions for a total of 100 points. Show all your work! **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number.

**You may not use a calculator on this exam. Please turn off and put away your cell phone.**

Do not write in this box.

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1. (5 points) Determine the largest interval in which the following initial value problem is certain to have a unique solution:

\[ y'' + (\sin 3t)y' + (\ln |t|)y = 0, \quad y(1) = -2, \quad y'(1) = -1. \]

(a) \((-\infty, \infty)\)
(b) \((1, \infty)\)
(c) \((0, \infty)\)
(d) \((-\infty, 0)\)

2. (5 points) Solve the following initial value problem:

\[ 2\sqrt{t}y' + y = 1, \quad y(1) = 0. \]

(a) \(y(t) = e^{-2\sqrt{t}} - 1\)
(b) \(y(t) = 1 - e^{1 - \sqrt{t}}\)
(c) \(y(t) = 1 - e^{-\sqrt{t}}\)
(d) \(y(t) = e^{\sqrt{t} - 1} - 1\)
3. (5 points) Find the values $\alpha$ and $\beta$ for which the following equation is exact:

$$3x^2y + \alpha x^5y^3 + (\beta x^2 + x^6y^2) y' = 0.$$ 

(a) No such $\alpha$ or $\beta$ exist.
(b) $\alpha = 1, \beta = 3$
(c) $\alpha = 6, \beta = 2$
(d) $\alpha = 2, \beta = 1$

4. (5 points) Let $y(t)$ be the solution to the initial value problem:

$$y'' - 4y' + 4y = 0, \quad y(0) = 4, \quad y'(0) = 7.$$ 

Find $\lim_{t \to \infty} y(t)$.

(a) 0
(b) $\infty$
(c) $-\infty$
(d) The solution oscillates and does not reach a limit.
5. (5 points) Suppose the characteristic equation of a sixth order constant coefficients homogeneous linear equation is

\[ 4(r - 2)(2 + r)^2(1 - r)^3 = 0. \]

Which function below is not a solution of the equation?

(a) \( y = 0 \)
(b) \( y = -t^3 e^t \)
(c) \( y = 3e^{2t} - 8te^{-2t} \)
(d) \( y = 2t^2 e^t + 5te^{-2t+4} \)

6. (8 points) Consider a certain mass-spring system described by the equation

\[ 3u'' + \gamma u' + ku = 0, \quad \gamma \geq 0, \quad k > 0. \]

Answer the following questions. Be sure to justify your answer.

(a) (2 points) Suppose \( k = 12 \). For what value(s) of \( \gamma \) would the system be critically damped?

(b) (2 points) Suppose \( \gamma = 6 \) and \( k = 9 \). Will any solution of the equation cross the equilibrium position more than once?

(c) (2 points) Suppose \( \gamma = 12 \) and \( k = 15 \). Find the quasi period of the system.

(d) (2 points) Suppose a force of \( F(t) = 49 \sin(\alpha t) \) is applied to the system, and given that \( \gamma = 0 \) and \( k = 75 \). What is the value(s) of \( \alpha \) if the system exhibits resonance?
7. (10 points) Determine whether each statement below is TRUE or FALSE.

(a) Suppose \( y = C_1y_1(t) + C_2y_2(t) \) is a general solution of a certain second order linear equation \( y'' + p(t)y' + q(t)y = 0 \), then their Wronskian \( W(y_1(t), y_2(t)) = 0 \).

(b) Suppose \( y_1(t) \) is a solution of a certain second order nonhomogeneous linear equation \( y'' + p(t)y' + q(t)y = g(t) \), then \( y_2(t) = Cy_1(t) \), where \( C \) is any constant, is always another solution of the same equation.

(c) The equation \( \sum_{k=1}^{n} t^k y^{(k)} = 0 \) is an \( n \)-th order linear equation.

(d) The equation \( x \, dx + y \, dy = 0 \) is not an exact equation.

(e) The constant solution \( y = 0 \) is the only solution of the initial value problem
\[ y'' + t^2 y = 0, \quad y(0) = 0, \quad y'(0) = 0. \]
8. (10 points) An object of mass 4 kg is moving along a straight line, propelled by a constant force of 7200 N. Suppose the object’s drag coefficient is 2 kg/m, and that the drag force is proportional to the square of the object’s velocity. The object is initially moving at a velocity of 10 m/s.

(a) (4 points) Write an initial value problem (i.e. give an equation and an initial condition) that describes the velocity of this object. You do NOT need to solve the problem.

(b) (2 points) Is the equation in (a) a linear equation?

(c) (2 points) Is the equation in (a) a separable equation?

(d) (2 points) Approximately how fast will the object be moving after a very long time?
9. (11 points) Solve the following initial value problem

\[(t^2 + 1)y' = \frac{1}{y - 4}, \quad y(0) = 1.\]

Give your answer in the **explicit** form.
10. (12 points) Consider the autonomous differential equation
\[ y' = -3y^2(y - 1)^3(y + 3). \]

(a) (2 points) Find all of its equilibrium solutions.

(b) (6 points) Classify the stability of each equilibrium solution. Be sure to provide a clear reason for your answer.

(c) (2 points) Suppose \( y(\pi) = 2 \), find \( \lim_{t \to \infty} y(t) \).

(d) (2 points) Suppose \( y(2017) = -3 \), find \( y(2016) \).
11. (10 points) Given that \( y_1(t) = t^2 \ln t \) is a known solution of the linear differential equation

\[
t^2 y'' - 3ty' + 4y = 0, \quad t > 0.
\]

Use reduction of order to find the general solution of the equation.
12. (14 points) Consider the second order nonhomogeneous linear equation
\[ y'' - 5y' + 6y = e^{3t} + t^2 - 4. \]

(a) (3 points) Find \( y_c(t) \), the solution of its corresponding homogeneous equation.

(b) (7 points) Find the general solution of the nonhomogeneous linear equation.

(c) (4 points) What is the form of particular solution \( Y \) that you would use to solve the following equation using the Method of Undetermined Coefficients? **DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.**

\[ y'' - 5y' + 6y = t^2 e^{3t} \sin t + 8te^{2t}. \]