MATH 251

Examination I

February 26, 2015

FORM B

Name:_____

Student Number:_____

Section:_____

This exam has 14 questions for a total of 100 points. Show all your work! **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

Do not write in this box.

1:	(10)
2 through 10:	(45)
11:	(10)
12:	(12)
13:	(10)
14:	(13)
Total:	

1. (10 points) Consider the following differential equation

$$(1 + y^2)y' + y = \sin(t)$$

(a) (2 points) What is the equation's order?

(b) (2 points) Is the equation **linear**?

(c) (2 points) Is the equation **separable**?

(d) (2 points) Is the equation **exact**?

(e) (2 points) By changing only 1 character or symbol in the original equation, it could be made into a new equation which is **autonomous**. Indicate this change.

2. (5 points) According to the Existence and Uniqueness Theorem, which of the initial value problems below is **not** guaranteed to have a unique solution?

(a) $ty' - t^2y = \sqrt{\sin t}, \qquad y(\frac{\pi}{2}) = \frac{-\pi}{2}.$

(b) $y' + 4y = \ln(2 - t), \qquad y(3) = 1.$

(c) $(t^2 - 4)y'' + (t + 3)y' + t^{-1}y = 5e^{-2t}, \qquad y(-3) = 0, \qquad y'(-3) = 2.$

(d) $y'' - \tan(t)y = 0, \qquad y(0) = \pi, \qquad y'(0) = 2.$

3. (5 points) What is a suitable integrating factor that can be used to solve the equation

$$ty' + (3t^2 - 2)y = e^{-t^2}, \qquad t > 0?$$

DO NOT solve this differential equation.

(a) $\mu(t) = e^{t^3 - 2t}$

(b) $\mu(t) = e^{3t - \frac{2}{t}}$

(c) $\mu(t) = -2te^{\frac{3}{2}t^2}$

(d) $\mu(t) = t^{-2}e^{\frac{3}{2}t^2}$

4. (5 points) A 300-gallon capacity tank initially contains 200 gallons of brine solution with a salt concentration of 1 lb/gal. Solution containing a salt concentration of 3 lb/gal enters at a rate of 2 gal/min and the well-stirred mixture is pumped out at 4 gal/min. Which initial value problem below describes the amount of salt $Q(t)$ in the tank at any time t before the tank is completely emptied?

(a) $\dfrac{dQ}{dt} = 2 - \dfrac{4Q}{200 - 4t}, \qquad Q(0) = 200.$

(b) $\dfrac{dQ}{dt} = 6 - \dfrac{4Q}{200 - 4t}, \qquad Q(0) = 300.$

(c) $\dfrac{dQ}{dt} = 6 - \dfrac{4Q}{200 - 2t}, \qquad Q(0) = 200.$

(d) $\dfrac{dQ}{dt} = 2 - \dfrac{4Q}{300 - 2t}, \qquad Q(0) = 300.$

5. (5 points) Consider the autonomous equation

$$y' = y^2(y+4)^3(25-y^2).$$

Which of the statements below regarding its equilibrium solutions is **true**?

(a) There are more than 4 equilibrium solutions.

(b) $y = -5$ is unstable.

(c) $y = -4$ is semistable.

(d) $y = 5$ is asymptotically stable.

6. (5 points) Consider the autonomous equation

$$y' = (y^2-4)(1-y)^2.$$

Suppose $y(-2) = \dfrac{3}{2}$. To what value will $y(t)$ approach after a very long time?

(a) -2

(b) 1

(c) $\dfrac{3}{2}$

(d) 2

7. (5 points) For what values of α and β will the following equation be exact?

$$4x^3 - \alpha x^2 y + y^3 - 2\beta + (\beta x^3 + 3xy^2)y' = 0.$$

(a) $\alpha = 3$, $\beta = 0$

(b) $\alpha = -3$, $\beta = 1$

(c) $\alpha = -3$, $\beta = -1$

(d) $\alpha = 3$, $\beta = 1$

8. (5 points) Which equation below has the property that all of its nonzero solutions diverge away from zero as $t \to \infty$?

(a) $y'' + 2y' + 2y = 0$

(b) $y'' + 2y' - 3y = 0$

(c) $y'' - 2y' + y = 0$

(d) $y'' - 3y' - 4y = 0$

9. (5 points) Suppose $y_1(t) = e^{3t+6}$ and $y_2(t) = 2$ are two solutions of a certain second order differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

Which of the following statements is **false**?

(a) $y(t) = e^{6t+3} - 2$ is another solution.

(b) $y(t) = e^{3t-6} + 5$ is another solution.

(c) $y(t) = 0$ is another solution.

(d) $y(t) = e^{3t}$ is another solution.

10. (5 points) Find the general solution of the fourth order linear equation

$$y^{(4)} + 8y' = 0.$$

[Hint: Recall the algebraic identity: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$]

(a) $y(t) = C_1 t + C_2 e^{2t} + C_3 e^{-t}\cos(\sqrt{3}t) + C_4 e^{-t}\sin(\sqrt{3}t)$

(b) $y(t) = C_1 + C_2 e^{2t} + C_3 e^{\sqrt{3}t} + C_4 e^{-\sqrt{3}t}$

(c) $y(t) = C_1 t + C_2 e^{-2t} + C_3 e^{\sqrt{3}t} + C_4 e^{-\sqrt{3}t}$

(d) $y(t) = C_1 + C_2 e^{-2t} + C_3 e^{t}\cos(\sqrt{3}t) + C_4 e^{t}\sin(\sqrt{3}t)$

11. (10 points) Consider the following list of differential equations:

$$
\begin{aligned}
&A. \quad u'' + 4u' + 13u = 0 \\
&B. \quad u'' - 4u' + 4u = 0 \\
&C. \quad u'' + 6u' + 6u = \cos(t) \\
&D. \quad u'' + 5u' - 4u = 2 \\
&E. \quad u'' + u = 0 \\
&F. \quad u'' + 4u = \sqrt{3}\sin(4t) \\
&G. \quad u'' + 6u' + 9u = 0 \\
&H. \quad u'' + 9u = 5\pi\cos(3t)
\end{aligned}
$$

Each of the equations above may or may not describe the displacement of a mass-spring system. Each question below has **exactly** one correct answer. The same equation may be reused to answer more than one question.

(a) Which equation describes a mass-spring system that is critically damped?

(b) Which equation describes a mass-spring system that is underdamped?

(c) Which equation describes a mass-spring system that is undergoing resonance?

(d) Which equation describes a mass-spring system that exhibits a simple harmonic motion?

(e) Which equation describes a mass-spring system whose motion crosses the equilibrium position at most once?

12. (12 points) Solve the following initial value problem. Give your answer in the explicit form.

$$y' = \frac{xe^{2x}}{y}, \qquad y(0) = -\frac{1}{2}.$$

13. (10 points) Given that $y_1(t) = t^2$ is a known solution of the linear differential equation

$$t^2y'' - 5ty' + 8y = 0, \qquad t > 0.$$

Use reduction of order to find the general solution of the equation.

14. (13 points) Consider the second order nonhomogeneous linear equation

$$y'' - 6y' - 7y = 8e^{-t} - 7t - 6.$$

(a) (3 points) Find $y_c(t)$, the solution of its corresponding homogeneous equation.

(b) (5 points) Find a particular function $Y(t)$ that satisfies the equation.

(c) (2 points) Write down the general solution of the equation.

(d) (3 points) What is the **form** of particular solution Y that you would use to solve the following equation using the Method of Undetermined Coefficients? **DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.**

$$y'' - 6y' - 7y = e^t(t^2 + 1)\sin(2t).$$