

MATH 251

Examination I

February 16, 2012

FORM A

Name: _____

Student Number: _____

Section: _____

This exam has 12 questions for a total of 100 points. Show all your work! **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

Do not write in this box.

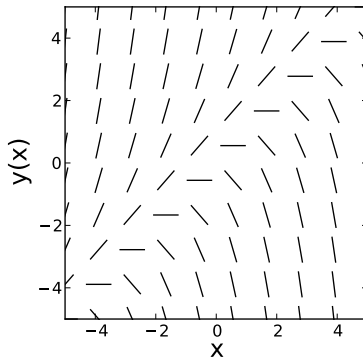
1 : _____
2
thru
8: _____
9: _____
10: _____
11: _____
12: _____
Total: _____

1. (10 points) Consider the differential equation

$$t^2 + t^3y + \sin(t)y - yy' = 0.$$

Answer the following questions. Briefly explain your answers.

- (a) (2 points) What is the equation's order?
- (b) (2 points) Is the equation **linear**?
- (c) (2 points) Is the equation **separable**?
- (d) (2 points) Is the equation **exact**?
- (e) (2 points) Is the equation **autonomous**?
2. (5 points) Determine the differential equation whose direction field is given below.



- (a) $y' = x + y$
- (b) $y' = x - y$
- (c) $y' = -x - y$
- (d) $y' = -x + y$

3. (5 points) Solve explicitly the initial value problem

$$y' = \frac{e^x}{2 + 2y} \quad y(0) = 1.$$

(a) $y(x) = -1 + \sqrt{e^x + 3}$

(b) $y(x) = -2 + \sqrt{e^x + 8}$

(c) $y(x) = 1 - \sqrt{e^x + 1}$

(d) $y(x) = 2 - \sqrt{e^x}$

4. (5 points) Consider the initial value problem

$$\sin(t)y' + \frac{1}{t-3}y = t^2, \quad y(1) = -2.$$

According to the Existence and Uniqueness Theorem, what is the largest interval in which a unique solution is guaranteed to exist?

(a) $(0, 3)$

(b) $(0, \pi)$

(c) $(-\infty, 3)$

(d) $(-\pi, \pi)$

5. (5 points) What is a suitable integrating factor that can be used to solve the equation

$$(1 + 2t^2)y' + ty = \sin(t)?$$

- (a) $\mu(t) = e^{-\frac{t^2}{2}}$
(b) $\mu(t) = \sqrt[4]{1 + 2t^2}$
(c) $\mu(t) = \sqrt{1 + 2t^2}$
(d) $\mu(t) = e^{t + \frac{2}{3}t^3}$

6. (5 points) A tank with a capacity of 400 liters originally contains 200 liters of hydrogen peroxide solution with concentration of 3 grams of H_2O_2 per liter of water. Additional hydrogen peroxide solution of concentration 6 grams/liter flows into the tank at a rate of 5 liters/min. The well mixed solution leaves the tank at a rate of 3 liters/min. Which of the initial value problems below best describes the amount of hydrogen peroxide, $H(t)$, in grams, that would be in the tank at time t , $0 < t < 100$?

- (a) $H' = 30 + \frac{3}{200 - 2t}H, \quad H(0) = 3.$
(b) $H' = 30 + \frac{3}{200 + 2t}H, \quad H(0) = 600.$
(c) $H' = 30 - \frac{3}{200 + 2t}H, \quad H(0) = 600.$
(d) $H' = 30 - \frac{3}{200 - 2t}H, \quad H(0) = 3.$

7. (5 points) What is the solution of the initial value problem

$$y'' - 4y' + 4y = 0, \quad y(3) = -1, \quad y'(3) = 2?$$

- (a) $y = -e^{2t-6} + 4te^{2t-6}$
- (b) $y = -13e^{2t-6} + 4te^{2t-6}$
- (c) $y = -e^{2t+6} - 4te^{2t+6}$
- (d) $y = 11e^{2t+6} + 4te^{2t+6}$

8. (5 points) Let $y(t)$ be the solution of the initial value problem

$$y'' + 2y' - 8y = 0, \quad y(0) = \alpha, \quad y'(0) = -2.$$

Suppose $\lim_{t \rightarrow \infty} y(t) = 0$, find the value of α .

- (a) $\alpha = \frac{1}{2}$
- (b) $\alpha = -1$
- (c) $\alpha = -4$
- (d) $\alpha = 8$

9. (14 points) Consider the equation

$$\frac{dy}{dx} = \frac{-e^y}{xe^y - \sin(y)}.$$

(a) (3 points) Rewrite it into the standard form of an exact equation.

(b) (3 points) Verify that it is an exact equation.

(c) (6 points) Find its general solution. You may leave your answer in implicit form.

(d) (2 points) Find the solution satisfying the initial condition $y(5) = 0$.

10. (13 points) Consider the autonomous differential equation

$$y' = 3y(y + 2)^2(4 - y).$$

(a) (3 points) Find all of its equilibrium solutions.

(b) (6 points) Classify the stability of each equilibrium solution. Justify your answer.

(c) (2 points) If $y(-2) = -1$, then what is $\lim_{t \rightarrow \infty} y(t)$?

(d) (2 points) If $y(10) = 0$, then what is $y(1000)$? Without solving the equation, briefly explain your conclusion.

11. (16 points) Suppose $y_1(t) = 4e^t$ and $y_2(t) = -2t$ are two solutions of a certain second order homogeneous linear equation

$$y'' + p(t)y' + q(t)y = 0.$$

- (a) (4 points) Find the Wronskian $W(y_1, y_2)(t)$.
- (b) (2 points) **True or false:** y_1 and y_2 form a set of fundamental solutions of this equation. Why or why not?
- (c) (3 points) Write down a general solution of the differential equation.
- (d) (3 points) Find the particular solution satisfying the initial conditions $y(0) = 8$ and $y'(0) = -1$.
- (e) (2 points) **True or false:** $y_3 = 0$ is also a solution of this equation. Why or why not?
- (f) (2 points) **True or false:** $y_4 = -8te^t$ is also a solution of this equation. Why or why not?

12. (12 points) Given that $y_1(t) = t^2$ is a known solution of the second order linear equation

$$t^2 y'' - 4ty' + 6y = 0, \quad t > 0.$$

Use reduction of order to find the general solution of the equation.