MATH 251

Examination I

February 25, 2010

FORM A

Name:____________________

Student Number:____________________

Section:____________________

This exam has 14 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

Do not write in this box.

1 / 2 :__________

3 thru 11:__________

12:__________

13:__________

14:__________

Total:__________

1. (12 points) Consider the differential equation

$$t^2 + 2t - 1 - 4y\,y' = 0.$$

Answer the following questions.

(a) (2 points) What is the order of this equation?

(b) (2 points) Is this equation linear?

(c) (2 points) Is this equation separable?

(d) (2 points) Is this equation exact?

(e) (4 points) Find its general solution. You may leave your answer in an implicit form.

2. (4 points) Find a suitable integrating factor that could be used to solve the equation below.

$$t\,y' - 4t^2 y = t^3 e^{9t} \tan(\pi t), \qquad t > 0.$$

3. (5 points) Suppose the velocity $v(t)$ of a speedboat is given by the equation

$$v' = 200 - \frac{1}{2}v^2, \qquad v \geq 0.$$

What is its limiting velocity, $\lim_{t\to\infty} v(t)$?

(a) 20

(b) 100

(c) 400

(d) ∞

4. (5 points) Consider the initial value problem

$$(t+1)(t-\pi)\, y' - \ln(t)\, y = \cos 3t, \qquad y(2) = 5.$$

Without solving the equation, what is the largest interval in which a unique solution is guaranteed to exist?

(a) $(0, \pi)$

(b) (π, ∞)

(c) $(-1, \pi)$

(d) $(-\infty, \infty)$

5. (5 points) A 500-liter mixing tank initially contains 300 liters of 3 grams/liter calcium bicarbonate solution. At $t = 0$, fresh water begins to flow into the vat at the rate of 4 liters/min. The thoroughly mixed content of the mixing tank is drawn off at the rate of 6 liters/min. Which of the initial value problems below best describes the quantity of calcium bicarbonate, $Q(t)$, that would be in the vat at time t, $0 < t < 150$?

(a) $Q' = 12 - \dfrac{3}{150-t}Q, \qquad Q(0) = 300.$

(b) $Q' = 12 - \dfrac{1}{50}Q, \qquad Q(0) = 300.$

(c) $Q' = -\dfrac{3}{150+t}Q, \qquad Q(0) = 900.$

(d) $Q' = -\dfrac{3}{150-t}Q, \qquad Q(0) = 900.$

6. (5 points) Which of the equations below is an exact equation whose (implicit) solution is given by

$$x^4y^4 + 2y = C?$$

(Hint: It is not actually necessary to solve any equation in order to answer this question.)

(a) $4x^4y^3 + 2 + 4x^3y^4\, y' = 0$

(b) $4x^3y^4 + (4x^4y^3 + 2)y' = 0$

(c) $\dfrac{1}{5}x^5y^4 + 2xy + (\dfrac{1}{5}x^4y^5 + y^2)y' = 0$

(d) $\dfrac{1}{5}x^4y^5 + y^2 + (\dfrac{1}{5}x^5y^4 + 2xy)y' = 0$

7. (5 points) Consider all the nonzero solutions of the equation

$$y'' + 2y' + 10y = 0.$$

As $t \to \infty$, they will

(a) all approach $-\infty$.

(b) all approach 0.

(c) some approach $+\infty$, some approach $-\infty$.

(d) reach no limits due to oscillation.

8. (5 points) Which pair of functions below cannot be a fundamental set of solutions?

(a) $3\cos 3t, \qquad 4\sin 3t$

(b) $-e^{4t}, \qquad 1$

(c) $e^{-5t}, \qquad 0$

(d) $2 + e^t, \qquad 2e^t - 1$

9. (5 points) Suppose $y_1(t) = t$ and $y_2(t) = e^{-t}$ are both solutions of the second order linear equation

$$y'' + p(t)\, y' + q(t)\, y = 0.$$

All of the functions below are also solutions of the same equation, EXCEPT

(a) $y = 0$

(b) $y = 5t - 2e^{-t}$

(c) $y = 9te^{-t}$

(d) $y = -10\pi\, t$

10. (5 points) Which equation below describes a mass-spring system that is undergoing resonance?

(a) $-2y'' - 8y = 7\cos 2t$.

(b) $y'' + 4y' + 4y = -16\sin 2t$.

(c) $y'' - 9y = 6\sin 3t$.

(d) $y'' + 16y = -3\cos 16t$.

11. (5 points) Consider the fourth order linear equation

$$y^{(4)} + 9y'' = 0.$$

What is its general solution?

(a) $y(t) = C_1 + C_2 \cos 3t + C_3 \sin 3t$

(b) $y(t) = C_1 t^2 + C_2 t\ \cos 3t + C_3 t\ \sin 3t$

(c) $y(t) = C_1 \cos 3t + C_2 \sin 3t + C_3 t\ \cos 3t + C_4 t\ \sin 3t$

(d) $y(t) = C_1 + C_2 t + C_3 \cos 3t + C_4 \sin 3t$

12. (13 points) Consider the autonomous differential equation

$$y' = y^4 - y^3 - y^2 + y = y\,(y+1)(y-1)^2.$$

(a) (3 points) Find all of its equilibrium solutions.

(b) (6 points) Classify the stability of each equilibrium solution. Justify your answer.

(c) (2 points) If $y(-2) = \frac{1}{2}$, what is $\lim_{t\to\infty} y(t)$?

(d) (2 points) If $y(4) = -1$, what is $y(t)$?

13. (12 points) Consider the nonhomogeneous second order linear equation of the form

$$y'' - 5y' + 4y = g(t).$$

(a) (3 points) Find $y_c(t)$, the solution of its corresponding homogeneous equation.

For each of the parts (b) through (d), choose from the list below the function that is the most suitable choice of the **form** of particular solution Y that you would use to solve the given equation using the Method of Undetermined Coefficients. **DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.**

A. $Y = (At + B)e^{-t} + Ce^{t}$

B. $Y = (At^2 + Bt)e^{-t} + Cte^{t}$

C. $Y = (At^2 + Bt)e^{-t} + Ce^{t}$

D. $Y = (At + B)e^{-t} + Cte^{t}$

E. $Y = Ae^{4t}\cos(2t) + Be^{4t}\sin(2t)$

F. $Y = Ae^{4t}\sin(2t)$

G. $Y = Ate^{4t}\cos(2t) + Bte^{4t}\sin(2t)$

H. $Y = (At^2 + Bt)e^{t}\cos(4t) + (Ct^2 + Dt)e^{t}\sin(4t)$

I. $Y = Ate^{t}\cos(4t)$

J. $Y = At^2e^{t}\cos(4t) + Bt^2e^{t}\sin(4t)$

K. $Y = (At + B)e^{t}\cos(4t) + (Ct + D)e^{t}\sin(4t)$

(b) (3 points) $y'' - 5y' + 4y = e^{4t}\sin 2t$

(c) (3 points) $y'' - 5y' + 4y = te^{-t} - 2e^{t}$

(d) (3 points) $y'' - 5y' + 4y = 7te^{t}\cos 4t$

14. (14 points) A mass-spring system is described by the initial value problem

$$4u'' + \gamma u' + 20u = 0, \qquad u(0) = 0, \qquad u'(0) = 4.$$

(a) (6 points) Suppose $\gamma = 8$. Find the real-valued particular solution of this initial value problem.

(b) (3 points) What is the quasi-period of this mass-spring system described in a)?

(c) (2 points) True or false: Some, but not all, nonzero solutions of this mass-spring system, with $\gamma = 8$ (regardless of initial conditions), will cross the equilibrium position more than once.

(d) (3 points) Find all the value(s) of γ that would make the system to be overdamped.