MATH 251

Examination I

February 28, 2008

Name:________________

Student Number:________________

Section:________________

This exam has 13 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.** The point value for each question is in parentheses to the right of the question number.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

Do not write in this box.

1:	
2:	
3:	
4:	
5:	
6:	
7:	
8:	
9:	
10:	
11:	
12:	
13:	
Total:	

1. (6 points) For each of the differential equations below state its order and whether it is linear or nonlinear.

	Equation	order	linear/nonlinear
(i)	$2y''' + ty'' - y' + e^t y = 0$		
(ii)	$y' = y^2 - y''$		
(iii)	$y' - \cos(2t)y = t^2 - e^{-5t}$		

2. (5 points) Which of the following initial value problems below is guaranteed to have a unique solution according to the appropriate Existence and Uniqueness Theorem?

   (a) $y' + \ln(t)\, y = 2, \qquad y(0) = 1$

   (b) $\cos(t)y' - 5y = t^2, \qquad y(\frac{\pi}{2}) = 0$

   (c) $y'' + 9y = \dfrac{1}{1 - e^t}, \qquad y(0) = 2, \qquad y'(0) = -2$

   (d) $y'' + (t^2 + 1)y' + t^3 y = 0, \qquad y(1) = -1, \qquad y'(1) = 0$

3. (5 points) Consider the second order linear equation with constant coefficients

$$ay'' + by' + cy = 0, \qquad a \neq 0.$$

Which of the following statements is **always** true?

(a) A unique solution exists satisfying the initial conditions $y(0) = \pi, y'(0) = \sqrt{\pi}$.

(b) Every solution is differentiable on $(-\infty, \infty)$.

(c) If y_1 and y_2 are any two linearly independent solutions, then $y = C_1\, y_1 + C_2\, y_2$ is a general solution of the equation.

(d) All of the above statements are always true.

4. (5 points) Let $y(t)$ be the solution of the initial value problem

$$y'' - y' - 6y = 0, \qquad y(0) = 4, \qquad y'(0) = \beta.$$

Suppose $\lim_{t\to\infty} y(t) = 0$, find the value of β.

(a) 0

(b) 4

(c) -8

(d) There is not a value of β that would make the limit zero.

5. (5 points) Let $y_1(t)$ and $y_2(t)$ be any two solutions of the second order linear equation

$$(t^2+6)y'' - ty' + e^{2t}y = 0.$$

In what general form must their Wronskian, $W(y_1, y_2)(t)$, appear?

(a) $C\sqrt{t^2+6}$

(b) $\dfrac{C}{\sqrt{t^2+6}}$

(c) $Ce^{\frac{1}{2}t^2}$

(d) $C(t^2+6)$

6. (5 points) Which equation below describes a mass-spring system that is undergoing resonance?

(a) $y'' + 9y = 2\cos 9t$.

(b) $y'' + 2y' + y = \cos t$.

(c) $y'' + 4y = 3\sin 2t$.

(d) $y'' + 16y = 4t$.

7. (5 points) Consider the fifth order linear equation

$$y^{(5)} - 2y^{(4)} + y^{(3)} = 0,$$

which has a characteristic equation $r^5 - 2r^4 + r^3 = r^3(r-1)^2 = 0$.
What is its general solution?

(a) $y(t) = C_1 + C_2 e^t + C_3 t e^t$

(b) $y(t) = C_1 t^2 + C_2 e^t + C_3 t e^t$

(c) $y(t) = C_1 + C_2 t + C_3 t^2 + C_4 e^t + C_5 e^{-t}$

(d) $y(t) = C_1 + C_2 t + C_3 t^2 + C_4 e^t + C_5 t e^t$

8. (5 points) Find the particular solution of the initial value problem

$$y' = \frac{4t^3 - 1}{y - 1}, \qquad y(0) = -1.$$

(a) $y(t) = 1 + \sqrt{2t^4 - 2t + 4}$

(b) $y(t) = 1 - \sqrt{2t^4 - 2t + 4}$

(c) $y(t) = -1 + \sqrt{2t^4 - 2t}$

(d) $y(t) = -1 - \sqrt{2t^4 - 2t}$

9. (13 points) Consider the autonomous differential equation

$$y' = y^4 + 2y^3 - 8y^2 = y^2\,(y-2)(y+4).$$

(a) (3 points) Find all equilibrium solutions.

(b) (6 points) Classify the stability of each equilibrium solution. Justify your answer.

(c) (2 points) If $y(-1) = 0$, what is $\lim_{t\to\infty} y(t)$?

(d) (2 points) If $y(0) = -1$, what is $\lim_{t\to\infty} y(t)$?

10. (13 points)

(a) (4 points) A 1000-gallon above-ground swimming pool is initially filled with 800 gallons of rain water. Water containing $5\,g/gal$ of chlorine flows into the pool at a rate of 6 gallons per minute. The well-mixed chlorine solution is pumped out at a rate of 3 gallons per minute. Let $Q(t)$ denote the amount of chlorine in the pool at any time $t > 0$, and before the pool eventually overflows. Write down an initial value problem (be sure to provide both a differential equation and an initial condition) that $Q(t)$ must satisfy. **DO NOT SOLVE THIS INITIAL VALUE PROBLEM.**

(b) (9 points) Let $P(t)$ denote the amount of salt in some mixing tank. If the initial value problem describing the amount of salt in the tank prior to overflow is

$$P'(t) = 8 - \frac{2P}{60 + 2t} \qquad P(0) = 4,$$

solve this initial value problem to find $P(t)$.

11. (10 points)

(a) (3 points) Verify that the differential equation

$$(4x^3y^4 + 2x) + (4x^4y^3 - 2)\, y' = 0$$

is an exact equation.

(b) (7 points) Find the particular solution of the above equation that satisfies the initial condition $y(1) = -1$. You may leave your answer in implicit form.

12. (11 points) Consider the nonhomogeneous second order linear equation of the form

$$y'' + 10y' + 25y = g(t).$$

(a) (3 points) Find its complementary solution, $y_c(t)$.

For each of next two parts, write down the correct choice of the **form** of particular solution that you would use to solve the given equation using the Method of Undetermined Coefficients. **DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.**

(b) (4 points) $y'' + 10y' + 25y = -te^{2t}\sin 5t - 2e^{-t}$

(c) (4 points) $y'' + 10y' + 25y = 4t^2e^{-5t} + t^2 - 7$

13. (12 points) A mass-spring system is described by the initial value problem

$$u'' + 4u' + 20u = 0, \qquad u(0) = 2, \qquad u'(0) = 0.$$

(a) (8 points) Find the real-valued particular solution of this initial value problem.

(b) (2 points) What is the quasi-frequency of this mass-spring system.

(c) (2 points) True or false: Every nonzero solution of this mass-spring system will cross the equilibrium position more than once.