

MATH 251  
Midterm Exam I  
Feb. 26, 2004

Name: \_\_\_\_\_  
Student Number: \_\_\_\_\_  
Instructor: \_\_\_\_\_  
Section: \_\_\_\_\_

This exam has **10** questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.**

**THE USE OF CALCULATORS IS NOT PERMITTED IN THIS EXAMINATION.**  
At the end of the examination, the booklet will be collected.

**Do not write in this box.**

1: _____
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10: _____
Total: _____

1. (5 points) Consider the initial value problem:

$$\frac{dy}{dt} = \frac{1}{y - 3t^2}, \quad y(t_0) = y_0$$

Give conditions on the values of  $t_0$  and  $y_0$  so that you are guaranteed to find a unique solution to the problem. Do not attempt to solve the equation for  $y(t)$ .

2. (12 points) For the autonomous equation  $y' = y^3 + 2y^2 - 3y$

a) Find all equilibrium solutions.

b) Determine the stability of each equilibrium solution you found in part (a). Justify your conclusions.

c) Let  $y(t)$  be the solution whose initial condition is  $y(0) = 4$ . What is the behavior of  $y(t)$  as  $t \rightarrow +\infty$  ?

d) Let  $y(t)$  be the solution whose initial condition is  $y(0) = -1$ . What is the behavior of  $y(t)$  as  $t \rightarrow +\infty$  ?

3. (10 points) Given that

$$y^2 e^{xy^2} + 4x^3 + (2xy e^{xy^2} + 2) \frac{dy}{dx} = 0, \quad y(0) = 2$$

a) Verify that the equation is exact.

b) Solve the initial value problem. Leave your answer in implicit form.

4. (15 points) For the equation

$$y'' - 4y' - 5y = 0$$

a) Find a fundamental pair of solutions.

b) Based on a), find the general solution.

c) Find the solution satisfying initial conditions  $y(0) = 2$ ,  $y'(0) = 3$ .

5. (a) (6 points) Solve the initial value problem:

$$y' = \frac{\cos t + 1}{e^y} \quad y(0) = 3.$$

- (b) (6 points) Solve the initial value problem

$$t^2 y' + ty = 2 \quad y(1) = 2.$$

6. Find general solutions to the following:

(a) (6 points)  $y'' + 2y' + 3y = 0$ .

(b) (6 points)  $y'' + 6y' + 9y = 0$ .

7. (6 points) Indicate whether each equation below is linear or non-linear.

(a)  $\sin(x) + e^x \frac{dy}{dx} = y^4$ .

(b)  $y'' + p(t)y + e^t = 0$ .

(c)  $y' = y^2 \cos(t)$ .

(d)  $y'' = \cos(y)$ .

(e)  $y''' + t^4 y' + t^3 y = 0$ .

8. (6 points) Indicate whether the two functions are linearly independent on the interval  $(-\infty, +\infty)$ :

(a)  $e^{2t}, e^{2t} + 2$

(b)  $\sin(2t), \sin(2t + 2\pi)$

(c)  $t + 1, 2t + 2$

9. (7 points) Given the linear ordinary differential equation  $ty'' - 4y' + \frac{4e^t}{t}y = 0$  and two fundamental solutions  $y_1(t), y_2(t)$  such that  $y_1(1) = 1$ ,  $y_1'(1) = 0$ ,  $y_2(1) = 2$  and  $y_2'(1) = 3$ , compute their Wronskian  $W(y_1(t), y_2(t))$  as a function of time, using Abel's Theorem. Use the initial condition to determine the constant of the Wronskian.

10. (15 points) A tank initially contains 120 liters of pure water. A salt solution with a concentration of  $\gamma$  grams/liter of salt enters the tank at a rate of 2 liters/min and the well-stirred mixture leaves the tank at the same rate. Find (in terms of  $\gamma$ ) an expression for the amount of salt in the tank at any time  $t$  and the limiting amount of salt in the tank as  $t$  approaches infinity.