MATH 251

Examination I

October 5, 2017

FORM A

Name:__________________

Student Number:__________________

Section:__________________

This exam has 13 questions for a total of 100 points. Show all your work! **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

Do not write in this box.

1 through 6:		(30)
7:		(8)
8:		(6)
9:		(12)
10:		(10)
11:		(10)
12:		(10)
13:		(14)
Total:		

1. (5 points) Consider the differential equation

$$y' = -2\sin(2y)$$

Which of following statements is true?

(I) The equation is separable, but not linear.

(II) The general solution is $y = \cos(2y) + C$.

(a) Only (I) is true.

(b) Only (II) is true.

(c) Both are true.

(d) Neither is true.

2. (5 points) Consider the initial value problem

$$t\frac{dy}{dt} + (t^2 - 1)y = \ln|t-5|, \qquad y(2) = 6.$$

According to the Existence and Uniqueness Theorem, what is the largest interval in which a unique solution is guaranteed to exist?

(a) $(0, 5)$

(b) $(-\infty, 5)$

(c) $(5, \infty)$

(d) $(1, \infty)$

3. (5 points) What is a suitable integrating factor that can be used to solve the equation

$$ty' + (2 - 4t)y = e^{-t}\cos(2t).$$

DO NOT solve this differential equation.

(a) $\mu(t) = e^{2t-2t^2}$

(b) $\mu(t) = e^{\frac{2}{t}-4}$

(c) $\mu(t) = t^{-4}e^{2t}$

(d) $\mu(t) = t^2e^{-4t}$

4. (5 points) Which equation below has the property that **some**, **but not all**, of its nonzero solutions converge to zero as $t \to \infty$?

(a) $y'' + 2y' + y = 0$

(b) $y'' + 4y' - 5y = 0$

(c) $y'' - 5y' + 6y = 0$

(d) $y'' + 4y = 0$

5. (5 points) Which pair of functions below can be a fundamental set of solutions for a second order homogeneous linear equation?

   (a) $y_1 = t^2 - 2t, \qquad y_2 = 8t - 4t^2$

   (b) $y_1 = 0, \qquad y_2 = \sin 2t - \cos 2t$

   (c) $y_1 = 3, \qquad y_2 = 3t$

   (d) $y_1 = e^{4t+1}, \qquad y_2 = e^{4t-9}$

6. (5 points) Find the general solution of the fifth order linear equation

$$y^{(5)} - 6y^{(3)} + 9y' = 0.$$

   (a) $C_1 + (C_2 + C_3 t)\sin(\sqrt{3}t) + (C_4 + C_5 t)\cos(\sqrt{3}t)$

   (b) $C_1 + C_2 e^{-\sqrt{3}t} + C_3 t e^{-\sqrt{3}t} + C_4 t^3 e^{-\sqrt{3}t} + C_5 t^4 e^{-\sqrt{3}t}$

   (c) $C_1 + C_2 e^{-\sqrt{3}t} + C_3 t e^{-\sqrt{3}t} + C_4 e^{\sqrt{3}t} + C_5 t e^{\sqrt{3}t}$

   (d) $C_1 + C_2 e^{-\sqrt{3}t} + C_3 e^{\sqrt{3}t} + C_4 \sin(\sqrt{3}t) + C_5 \cos(\sqrt{3}t)$

7. (8 points) True or false:

   (a) $(t^2+1)y' = e^{-3t}\arctan(t)\,y$ is a first order equation that is both linear and separable.

   (b) Every separable equation $M(t) + N(y)\frac{dy}{dt} = 0$ is also an exact equation.

   (c) Every autonomous equation $y' = f(y)$ is also a linear equation.

   (d) Given that y_1 and y_2 are both solutions of $y'' + 10y = e^{-t}$. Then $y_3 = y_1 + y_2$ is also a solution of the same equation.

8. (6 points) A tank with capacity 2000 liters initially contains 1000 liters of water with 20 kg of dissolved salt. Water containing $0.5 + e^{-t}$ kg/liter of salt enters at a rate of 3 liters/min, and the well-stirred mixture flows out of the tank at a rate of 2 liters/min. Let $Q(t)$ denote the amount of salt in the tank at any time t. Compose (but do NOT solve) an initial value problem that accurately describes $Q(t)$, for $0 < t < 1000$.

9. (12 points) Consider the autonomous differential equation

$$y' = (e^y - 1)(y - 3)(5 - y).$$

(a) (2 points) Find all equilibrium solutions of this equation.

(b) (6 points) Classify the stability of each equilibrium solution. Justify your answer.

(c) (2 points) Suppose $y(9) = \alpha$, and $\lim_{t\to\infty} y(t) = 0$. Find all possible value(s) of α.

(d) (2 points) (Circle the correct answer.) Which one of the two initial conditions below will result in a constant solution to the equation? What is the resulted solution?

$$y(3) = 4, \qquad \textit{or} \qquad y(-2) = 5.$$

10. (10 points) Consider the following list of differential equations:

$$\begin{array}{ll} A. & u'' + 2u' + 5u = 0 \\ B. & u'' - 4u' + 13u = 0 \\ C. & u'' + 17u = 0 \\ D. & u'' + u = \sin(2t) \\ E. & u'' + 4u' + 4u = \sin(2t) \\ F. & u'' + 9u = -2\cos(3t) \\ G. & u'' + 6u' + 8u = 0 \end{array}$$

Each of the equations above may or may not describe the displacement of a mass-spring system. Each question below has **exactly** one correct answer. The same equation may be reused to answer more than one question.

(a) Which equation describes a mass-spring system that is undergoing resonance?

(b) Which equation describes a mass-spring system that exhibits a simple harmonic motion?

(c) Which equation describes a mass-spring system whose motion crosses the equilibrium position at most once?

(d) Which equation describes a mass-spring system that is underdamped?

(e) Which equation describes a mass-spring system that is overdamped?

11. (10 points) Consider the differential equation

$$(2x^2 - 6y^2)y' + 4xy - 2e^{2x} = 0.$$

(a) (3 points) Verify that it is an exact equation.

(b) (7 points) Find the solution of this equation satisfying $y(0) = -1$. You may leave your answer in an implicit form.

12. (10 points) Given that $y_1(t) = (t+4)^3$ is a known solution of the linear differential equation

$$(t+4)^2 y'' - 5(t+4)y' + 9y = 0, \qquad t > -4.$$

Use reduction of order to find the general solution of the equation.

13. (14 points) Consider the second order nonhomogeneous linear equation

$$y'' - 7y' + 12y = 10\sin t + 12t + 5.$$

(a) (3 points) Find $y_c(t)$, the solution of its corresponding homogeneous equation.

(b) (7 points) Find a particular function $Y(t)$ that satisfies the equation.

(c) (1 point) Write down the general solution of the equation.

(d) (3 points) What is the **form** of particular solution Y that you would use to solve the following equation using the Method of Undetermined Coefficients? **DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.**

$$y'' - 7y' + 12y = 3t^2e^{4t} - te^{3t}\sin t.$$