

MATH 251
Examination I
October 10, 2012
FORM A

Name: _____
Student Number: _____
Section: _____

This exam has 13 questions for a total of 100 points. Show all your work! **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE AND ALL OTHER MOBILE DEVICES.

Do not write in this box.

1
through
8: _____
9: _____
10: _____
11: _____
12: _____
13: _____
Total: _____

1. (5 points) Which of the equations below is a first order linear ordinary differential equation?

(a) $y' + e^{-t}y = \cos(\frac{1}{t})$

(b) $y'' = e^t \sin 2y$

(c) $(y')^2 + y = 0$

(d) $y'' - t \ln(t)y' + e^{5t}y = 6$

2. (5 points) What is a suitable integrating factor that can be used to solve the equation

$$2t^2y' + 8ty = \cos(3t)?$$

(a) $\mu(t) = -\frac{1}{t^4}$

(b) $\mu(t) = 4t$

(c) $\mu(t) = -t^2$

(d) $\mu(t) = t^4$

3. (5 points) Find the value of b for which the following equation is exact:

$$2e^x + x^5 + 3x^2y + (\sin y + bx^3 + \frac{1}{y^2 + 1})\frac{dy}{dx} = 0.$$

- (a) $b = 0$
 - (b) $b = 3$
 - (c) $b = -1$
 - (d) $b = 1$
4. (5 points) Consider the initial value problem

$$(t^2 + t)y' + \frac{1}{(t - 4)}y = e^{2t}, \quad y(2) = 3\pi.$$

According to the Existence and Uniqueness Theorem, what is the largest interval in which a unique solution is guaranteed to exist?

- (a) $(4, \infty)$
- (b) $(-\infty, 4)$
- (c) $(0, 4)$
- (d) $(1, 4)$

5. (5 points) A 800-gallon tank initially contains 500 gallons of water and 30 pounds of salt dissolved in it. Water enters the tank at the rate of 3 gal/min with concentration 4 lb/gal of salt in it. The well-mixed solution leaves the tank at the rate of 1 gal/min. Which of the initial value problems below models the change of the amount of salt $Q(t)$ inside the tank during the time interval $0 \leq t \leq 150$?

(a) $Q'(t) = 3 - \frac{Q(t)}{800}$ $Q(0) = 30$

(b) $Q'(t) = 12 - \frac{Q(t)}{500 - 2t}$ $Q(0) = 800$

(c) $Q'(t) = 12 - \frac{Q(t)}{500 + 2t}$ $Q(0) = 30$

(d) $Q'(t) = 12 - \frac{Q(t)}{500 + t}$ $Q(0) = 500$

6. (5 points) Let $y_1(t)$ and $y_2(t)$ be any two solutions of the second order linear equation

$$y'' + 2 \tan(t)y' + t^3 y = 0.$$

What is the general form of their Wronskian, $W(y_1, y_2)(t)$?

(a) $C \cos^2 t$

(b) $C e^{\sec^2 t}$

(c) $C e^{2 \sin t}$

(d) $C \sec^2 t$

7. (5 points) Which pair of functions below can be a fundamental set of solutions?

(a) $0, \quad 5te^{2t}$

(b) $\sin 3t, \quad -\sin 3t$

(c) $6e^{-t}, \quad -6e^t$

(d) $4t - \cos t, \quad 3 \cos t - 12t$

8. (5 points) Suppose $y_1(t) = -t^2$ and $y_2(t) = 3t^2 \ln(t)$ are both solutions of the second order linear equation

$$y'' + p(t)y' + q(t)y = 0, \quad t > 0.$$

Which statement below is FALSE?

(a) $y = 2t^2 - 10t^2 \ln(t)$ is also a solution of the equation.

(b) $W(y_1, y_2)(t) = 0$

(c) $y = 0$ is also a solution of the equation.

(d) $y = 1$ can never be a solution of the equation.

9. (10 points) Solve the initial value problem

$$y' = \frac{2 - 3x^2 + 8x^3}{2 + 2y} \qquad y(1) = -2.$$

Give your solution in the explicit form.

10. (14 points) A mass-spring system is described by the equation

$$5u'' + \gamma u' + ku = F(t).$$

- (a) (2 points) Suppose the mass originally stretched the spring $2m$ to reach its equilibrium position. What is the spring constant k ? (Assume $g = 10m/s^2$ to be the gravitational constant.)
- (b) (2 points) Suppose $k = 45$. For what value(s) of γ would this system be critically damped?
- (c) (2 points) Suppose $\gamma = 0$ and $k = 400$. What is the natural frequency of this system?

A second mass-spring system is described by the initial value problem

$$2u'' + 12u' + 20u = 0, \quad u(0) = 0, \quad u'(0) = 2.$$

- (d) (6 points) Find the position of the mass when $t = \frac{3\pi}{2}$.

- (e) (2 points) What is the quasi-frequency of this system?

11. (13 points) Consider the autonomous differential equation

$$y' = (y - 1)(y - 2)(y + 3)^2.$$

(a) (3 points) Find all of its equilibrium solutions.

(b) (6 points) Classify the stability of each equilibrium solution. Clearly explain how you have obtained your answer.

(c) (2 points) If $y(-3) = -1$, then what is $\lim_{t \rightarrow \infty} y(t)$?

(d) (2 points) If $y(500) = 2$, then what is $y(2500)$? Without solving the equation, briefly explain your conclusion.

12. (10 points) Consider the second order linear equation

$$y'' - 8y' + 16y = 0.$$

(a) (3 points) Find the general solution of the equation.

(b) (5 points) Find the solution satisfying the initial conditions $y(735) = 2$, $y'(735) = -1$.

(c) (2 points) Find $\lim_{t \rightarrow \infty} y(t)$.

13. (13 points) Consider the second order nonhomogeneous linear equation

$$y'' - 2y' - 3y = e^{-t} + 3.$$

(a) (3 points) Find $y_c(t)$, the solution of its corresponding homogeneous equation.

(b) (7 points) Find its general solution.

(c) (3 points) What is the **form** of particular solution Y that you would use to solve the following equation using the Method of Undetermined Coefficients? **DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.**

$$y'' - 2y' - 3y = 7te^{-t} \sin 2t.$$