Math 251H-01

Midterm Exam 2

April 4, 2016

Name:______________________________________________

Instructions: Clearly answer each of the questions. Box your answers. Partial credit will be awarded based on the clarity and correctness of your explanation of each solution. Use the backs of pages for your scratch work (or overflow). There are 7 problems. Make sure you have all 12 pages.

A table of Laplace transforms is provided on page 12.

<table>
<thead>
<tr>
<th>Problem</th>
<th>out of</th>
<th>score</th>
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</thead>
<tbody>
<tr>
<td>1</td>
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<td>7</td>
<td>12</td>
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<tr>
<td>Total</td>
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</table>
1. **(6 points)** Use the definition of the Laplace transform to determine \( \mathcal{L}\{f\} \), if it exists. If it does not exist, state why.

(a)

\[
f(t) = \begin{cases} 
  e^{-t}, & 0 \leq t < 5 \\
  -1, & t \geq 5 
\end{cases}
\]

(b)

\[
f(t) = \begin{cases} 
  e^{-t^2}, & 0 \leq t < 5 \\
  e^{t^2}, & t \geq 5 
\end{cases}
\]
2. (5 points) The mixing tank shown below initially holds 500L of a brine solution with a salt concentration of 0.2 kg/L. For the first 10 min of operation, valve A is open, adding 12 L/min of brine containing a 0.4 kg/L salt concentration. After 10 minutes, valve B is switched in, adding a 0.6 kg/L concentration at 12 L/min. The exit valve C removes 12 L/min, thereby keeping the volume constant.

Write down the initial value problem for the mass of salt in the tank at time \( t \) in terms of unit step functions.

Do not solve.
3. (8 points) Use the method of Laplace transforms to solve the integro-differential equation

\[
\begin{align*}
    y'(t) + 5 \int_0^t y(t - \tau) e^{A\tau} d\tau &= 0 \\
y(0) &= 1.
\end{align*}
\]
4. (12 points) Consider the initial value problem

\[
\begin{align*}
y'' + 4y &= -\delta(t - \pi) \\
y(0) &= 0, \ y'(0) &= 1.
\end{align*}
\]

(a) Interpret as a mass-spring system. Feel free sketch if it helps you explain!

(b) Use the method of Laplace transforms to determine \( y(t) \).
(c) Sketch $y(t)$. Don’t forget to label your axes.
5. (7 points) Consider the system
\[
\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ \alpha & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]
Determine the value(s) or range(s) in \( \alpha \) where the critical point (0,0) is an:

\begin{align*}
\text{___________} & \quad \text{unstable node} & \quad \text{___________} & \quad \text{asymptotically stable node} \\
& & & \quad \text{(proper or improper)} \\
\text{___________} & \quad \text{stable center} & \quad \text{___________} & \quad \text{asymptotically stable, spiral point} \\
\text{___________} & \quad \text{unstable, saddle point} & \quad \text{___________} & \quad \text{unstable spiral point}
\end{align*}

\textbf{Not all the categories apply!}
6. (10 points) (a) Find the general solution of the system of linear equations

\[ \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6e^{-2t} \\ 8 \end{pmatrix} \]

given that the homogeneous solution is

\[ \vec{x}_h = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}. \]

(b) Find the solution satisfying \( x(0) = y(0) = 0 \).
7. (12 points) The second order differential equation for the undamped, unforced mass–spring oscillator

\[ mx'' + kx = 0 \]

for \( x(t) \) the mass distance in meters from equilibrium, can be re-written as

\[
\begin{cases}
  x'(t) = v \\
  v'(t) = -\frac{k}{m}x
\end{cases}
\Rightarrow
\begin{cases}
  x'(t) = v \\
  v'(t) = -4x
\end{cases}
\]

where \( v(t) = x'(t) \) is the spring’s velocity, assuming the mass \( m = 1 \) kg, and spring constant \( k = 4 \) N/m.

(a) Find the fundamental solution set.
(b) Sketch the phase portrait, identifying fundamental solutions and critical point.

(c) Assume that at $t = 0$, the mass is at 3 m from equilibrium, at rest. Add to your phase portrait the solution for the mass’ motion for $0 \leq t < \infty$. Label the solution.

(d) Sketch the spring’s position as a function of time, $x$ vs $t$, starting from the initial conditions given in (b). Don’t forget to label your axes! **Hint:** You do not have to find the solution to the IVP, the graph can be inferred from the phase portrait.

(e) **Interpret:** what happens to the position and velocity of the mass as $t \to \infty$ assuming the initial conditions given in (b)?
### Elementary Laplace transforms

<table>
<thead>
<tr>
<th>$f(t) = \mathcal{L}^{-1}{F(s)}$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
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</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$\frac{1}{s}, , s &gt; 0$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}, , s &gt; a$</td>
</tr>
<tr>
<td>$t^n, , n$ a positive integer</td>
<td>$\frac{n!}{s^{n+1}}, , s &gt; 0$</td>
</tr>
<tr>
<td>$t^p, , p &gt; 1$</td>
<td>$\frac{\Gamma(p+1)}{s^{p+1}}, , s &gt; 0$</td>
</tr>
<tr>
<td>$\sin(at)$</td>
<td>$\frac{a}{s^2+a^2}, , s &gt; 0$</td>
</tr>
<tr>
<td>$\cos(at)$</td>
<td>$\frac{s}{s^2+a^2}, , s &gt; 0$</td>
</tr>
<tr>
<td>$\sinh(at)$</td>
<td>$\frac{a}{s^2-a^2}, , s &gt;</td>
</tr>
<tr>
<td>$\cosh(at)$</td>
<td>$\frac{s}{s^2-a^2}, , s &gt;</td>
</tr>
<tr>
<td>$e^{at}\sin(bt)$</td>
<td>$\frac{b}{(s-a)^2+b^2}, , s &gt; a$</td>
</tr>
<tr>
<td>$e^{at}\cos(bt)$</td>
<td>$\frac{s-a}{(s-a)^2+b^2}, , s &gt; a$</td>
</tr>
<tr>
<td>$t^n e^{at}, , n$ a positive integer</td>
<td>$\frac{n!}{(s-a)^{n+1}}, , s &gt; a$</td>
</tr>
<tr>
<td>$u_c(t)$</td>
<td>$\frac{e^{-cs}}{s}, , s &gt; 0$</td>
</tr>
<tr>
<td>$u_c(t)f(t-c)$</td>
<td>$e^{-cs}F(s)$</td>
</tr>
<tr>
<td>$e^{ct}f(t)$</td>
<td>$F(s-c)$</td>
</tr>
<tr>
<td>$f(ct)$</td>
<td>$\frac{1}{c}F\left(\frac{s}{c}\right)$</td>
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<tr>
<td>$\int_0^t f(t-\tau)g(\tau),d\tau$</td>
<td>$F(s)G(s)$</td>
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<tr>
<td>$\delta(t-c)$</td>
<td>$e^{-cs}$</td>
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<tr>
<td>$f^{(n)}(t)$</td>
<td>$s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$</td>
</tr>
<tr>
<td>$(-t)^n f(t)$</td>
<td>$F^{(n)}(s)$</td>
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