Math 251H-01

Midterm Exam 1

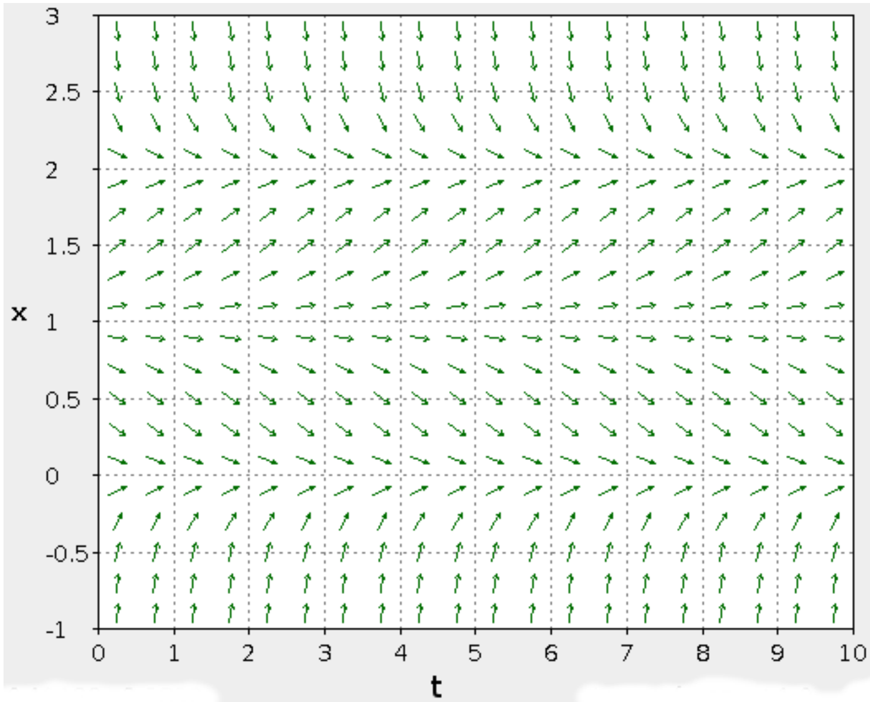
February 25, 2016

Name:____

Instructions: Clearly answer each of the questions. Box your answers. Partial credit will be awarded based on the clarity and correctness of your explanation of each solution. Use the backs of pages for your scratch work (or overflow). There are **9** problems. Make sure you have all **11** pages.

Problem	**out of**	**score**
1	5	
2	6	
3	5	
4	8	
5	8	
6	8	
7	6	
8	6	
9	8	
Total	60	

1. (**5** points) Consider the direction field



(a) What is the differential equation associated with this direction field? (circle the correct one)

(i) $x'(x) = x(x+1)(x+2)$

(ii) $x'(t) = x(1-x)(x-2)$

(iii) $x'(t) = -x(x+1)(x+2)$

(iv) $x(t) = x(x-1)(x-2)$

(b) On the direction field, sketch & label an approximate solution if $x(0) = 0.5$.

(c) On the direction field, sketch & label an approximate solution if $x(0) = 1.5$.

(d) List all asymptotically stable solutions, if any (you do not need to justify your answer).

(e) List all unstable solutions, if any (you do not need to justify your answer).

2. (**6** points) Solve the initial value problem

$$\begin{cases} \sqrt{y}\,dx + (1+x)\,dy = 0, \quad -1 < x < e^2 - 1 \\ y(0) = 1 \end{cases}$$

3. (**5** points) Find the most general function $M(x,y)$ so that the equation

$$M(x,y)\,dx + (\sin x \cos y - xy - e^y)\,dy = 0$$

is exact.

4. (**8** points) A brine solution of salt flows at a constant rate of 4 L/min into a large tank that initially held 100 L of pure water. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 3 L/min. Assume that the concentration of salt in the brine entering the tank is 0.2 kg/L.

Determine the **concentration** of salt in the tank after t min.

5. (**8** points) The function $y_1(x) = x$ solves the differential equation

$$x^2 y'' - xy' + y = 0$$

on the interval $x > 0$.

(a) Find a second linearly independent solution $y_2(x)$.

(b) Prove that $y_1(x)$ and your $y_2(x)$ are indeed linearly independent.

(c) Write down the general solution of $x^2 y'' - xy' + y = 0$.

6. (**8** points) Find the general solution of

$$x'' + 4x' + 4x = \frac{e^{-2t}}{t}, \;\; t > 0$$

7. (**6** points) Find the fundamental solution set of

$$y^{(5)} + 16y' = 0.$$

8. (**6** points) I've set up an LCR circuit as a crystal radio with a capacitor of 0.2×10^{-6} F and a resistor of $5 \times 10^3\,\Omega$. I want to use my radio to pick up a frequency of 1000 kHz (i.e., the forcing term is $F\cos(1000000t)$). What value should I choose for the inductor to pick up the radio signal most clearly? Why?

9. (**8** points) See following page for the problem.

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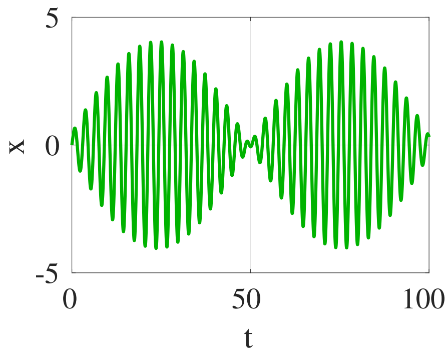
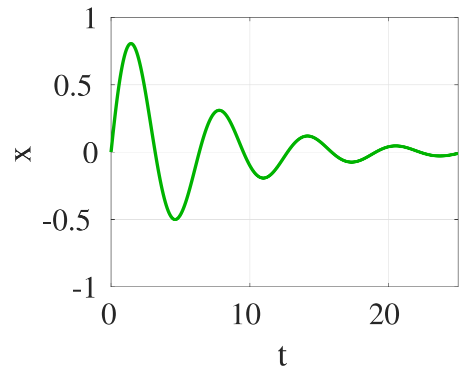
<tr><td></td><td>
</td><td></td><td>
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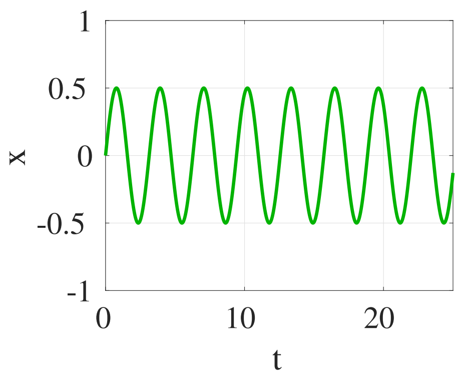
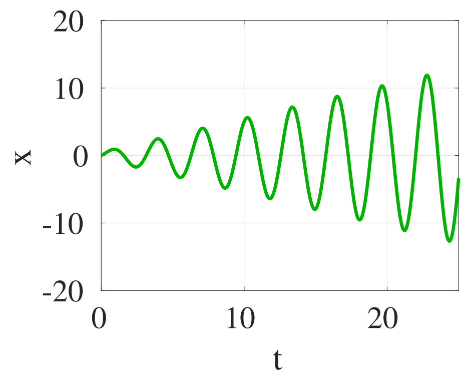
<tr><td></td><td>
</td><td></td><td>
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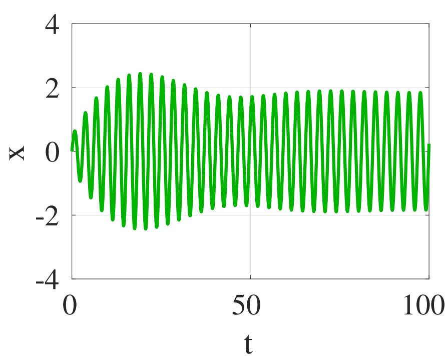
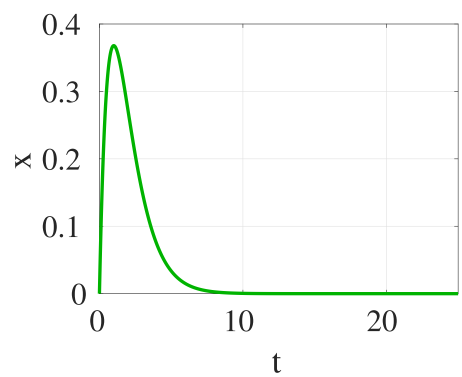
<tr><td></td><td>
</td><td></td><td>
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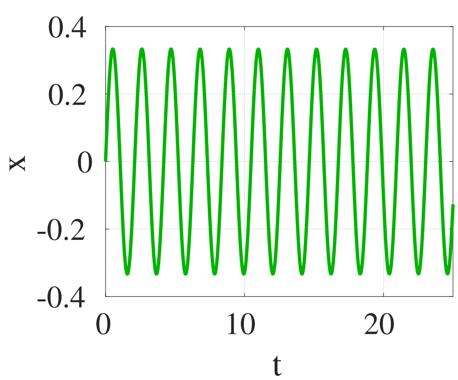
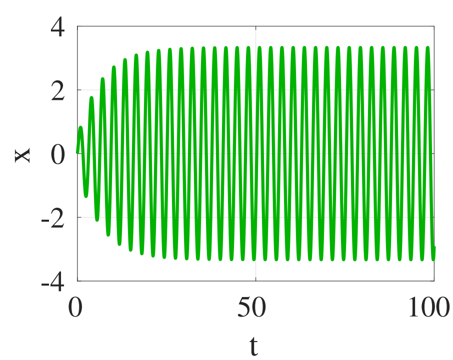
<tr><td></td><td>
</td><td></td><td>
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Consider the forced mass-spring system, which can be described by:

$$x''(t) + bx'(t) + kx(t) = F\cos(\beta t), \quad x(0) = 0, \quad x'(0) = 1,$$

where $x(t)$ is the distance of the mass from equilibrium at time t, b the damping, and k the spring constant. The forcing has amplitude F and frequency β. Assume the mass is 1 kg.
On the previous page you will find plots of different solutions $x(t)$ for various values of b, k, F, and β, corresponding in no particular order to:

A $b = 0.3$, $k = 1$, $F = 0$, β not relevant $\Rightarrow$
$x''(t) + 0.3x'(t) + x(t) = 0, \quad x(0) = 0, \quad x'(0) = 1.$

B $b = 0.3$, $k = 4$, $F = 2$, $\beta = 2 \Rightarrow$
$x''(t) + 0.3x'(t) + 4x(t) = 2\cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$

C $b = 0$, $k = 9$, $F = 0$, β not relevant $\Rightarrow$
$x''(t) + 9x(t) = 0, \quad x(0) = 0, \quad x'(0) = 1.$

D $b = 0$, $k = 4.5$, $F = 1$, $\beta = 2 \Rightarrow$
$x''(t) + 4.5x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$

E $b = 2$, $k = 1$, $F = 0$, β not relevant $\Rightarrow$
$x''(t) + 2x'(t) + x(t) = 0, \quad x(0) = 0, \quad x'(0) = 1.$

F $b = 0$, $k = 4$, $F = 2$, $\beta = 2 \Rightarrow$
$x''(t) + 4x(t) = 2\cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$

G $b = 0$, $k = 4$, $F = 0$, β not relevant $\Rightarrow$
$x''(t) + 4x(t) = 0, \quad x(0) = 0, \quad x'(0) = 1.$

H $b = 0.1$, $k = 4.5$, $F = 2$, $\beta = 2 \Rightarrow$
$x''(t) + 0.1x'(t) + 4.5x(t) = 2\cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$

Match the differential equation with its solution's plot by writing the appropriate letter in the box to the *left* of the plot, on the previous page.