There are 12 multiple-choice questions, 8 T/F questions, and 3 partial credit questions on this exam, for a total of 150 points. In order to obtain full credit for the partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.

The use of calculators is not permitted in this examination.

There are 16 problems on 11 pages, including this one. Check your booklet now.

The area below is for the instructor’s use.

M.C. __________________ (72)
T/F __________________ (24)
14. __________________ (18)
15. __________________ (18)
16. __________________ (18)
Total __________________ (150)
1. (6 pts.) The Existence and Uniqueness Theorem guarantees that the solution to
\[ ty' + \sqrt{t + 4} \ y = \frac{1}{t - 5} \quad y(-1) = 1 \]
is valid on

a) \((-4, 0)\)

b) \((-\infty, 0)\)

c) \((-4, 5)\)

d) \((-\infty, \infty)\)

2. (6 pts) The autonomous equation
\[ y' = y^2 - 4 \]
has

a) An asymptotically stable equilibrium solution at \(y = 2\), and an unstable equilibrium solution at \(y = -2\);

b) An asymptotically stable equilibrium solution at \(y = -2\), and an unstable equilibrium solution at \(y = 2\);

c) Two asymptotically stable equilibrium solutions;

d) Two unstable equilibrium solutions.
3. (6 pts) The general solution of \( y'' + 2y' + y = 0 \) is

a) \( y(t) = C_1 e^{-t} \),

b) \( y(t) = C_1 e^t + C_2 e^{-t} \),

c) \( y(t) = C_1 \cos t + C_2 \sin t \),

d) \( y(t) = C_1 e^{-t} + C_2 te^{-t} \).

4. (6 pts) Let \( y(t) \) be the solution of the initial value problem

\[ y'' - 3y' - 4y = 0 \quad y(0) = 2, \quad y'(0) = \alpha. \]

Suppose \( \lim_{t \to \infty} y(t) = 0 \), what is the value of \( \alpha \)?

a) \(-2\)

b) 0

c) 3

d) It cannot be determined.
5. (6 pts) When solving the following second order nonhomogeneous linear equation using the Method of Undetermined Coefficients, what form does its particular solution $Y(t)$ have?

$$y'' + 2y' + y = 2e^{-t} - t \sin 3t$$

a) $Ae^{-t} + Bt \cos 3t + Ct \sin 3t,$  
b) $Ate^{-t} + (Bt^2 + Ct) \cos 3t + (Dt^2 + Et) \sin 3t,$  
c) $At^2 e^{-t} + Bt \cos 3t + Ct \sin 3t,$  
d) $At^2 e^{-t} + (Bt + C) \cos 3t + (Dt + E) \sin 3t.$

6. (6 pts) Which equation below describes a mass-spring system that is undergoing resonance?

a) $y'' + 4y = 3 \cos 4t,$  
b) $y'' + y = 4 \sin t,$  
c) $y'' + 4y' + 4y = 2 \sin 2t,$  
d) $y'' + 2y' + 9y = 5 \cos 3t.$
7. (6 pts.) A mass of 3 kg stretches a spring 3 m. The system has a damping constant of $\gamma = 5$. Suppose the mass is initially set in motion from its equilibrium position with a downward velocity of $6 \frac{m}{s}$. Which of the following is the initial value problem describing the motion of this system? Use $g = 10 \frac{m}{s^2}$ as the gravitational constant.

a) $3u'' + 5u' + 3u = 0$, $u(0) = 6$, $u'(0) = 0$;
b) $3u'' + 5u' + 10u = 0$, $u(0) = 0$, $u'(0) = 6$;
c) $3u'' + 3u' + 5u = 0$, $u(0) = 0$, $u'(0) = 6$;
d) $3u'' + 5u' + 30u = 0$, $u(0) = 6$, $u'(0) = 0$.

8. (6 pts.) The inverse Laplace transform of $F(s) = \frac{2s - 1}{s^2 + 2s + 5}$ is

a) $2e^t \cos 2t - e^t \sin 2t$;
b) $2e^{-t} \cos 2t - e^{-t} \sin 2t$;
c) $2e^{2t} \cos t - \frac{1}{2} e^{2t} \sin t$;
d) $2e^{-t} \cos 2t - \frac{3}{2} e^{-t} \sin 2t$. 
9. (6 pts.) The piecewise continuous function

\[ f(t) = \begin{cases} 
  t - 1, & 0 \leq t < 4 \\
  2, & 4 \leq t 
\end{cases} \]

is equivalent to

a) \( f(t) = t - 1 + 2u_4(t) \),
b) \( f(t) = t - 1 + u_4(t)(t + 1) \),
c) \( f(t) = t - 1 + u_4(t)(3 - t) \),
d) \( f(t) = t + 3u_4(t) \).

10. (6 pts.) The Laplace transform of \( f(t) = u_1(t)e^{-3t} \) is

a) \( e^{-s} \frac{1}{s(s + 3)} \),
b) \( e^{-s} \frac{1}{s + 3} \),
c) \( 3e^{-s} \frac{1}{s + 3} \),
d) \( e^{-s-3} \frac{1}{s + 3} \).
11. (6 pts.) Which system of first order linear equations below is equivalent to the second order linear equation

\[ y'' + y' - 2y = 0 \]

a) \[
\begin{align*}
x_1' &= x_2 \\
x_2' &= -2x_1 + x_2
\end{align*}
\]

b) \[
\begin{align*}
x_1' &= x_1 \\
x_2' &= x_1 - 2x_2
\end{align*}
\]

c) \[
\begin{align*}
x_1' &= x_2 \\
x_2' &= 2x_1 - x_2
\end{align*}
\]

d) \[
\begin{align*}
x_1' &= -2x_1 + x_2 \\
x_2' &= x_1
\end{align*}
\]

12. (6 pts.) The critical point at (0, 0) of the linear system

\[
x' = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix} x
\]

is

a) An unstable node,

b) An asymptotically stable node,

c) An unstable saddle point,

d) A stable center.
13. (24 pts., 3 pts. each) True or false:

a) T F The equation \( y''' + t^2 y' - 5y = e^{2t} \) is an example of a third order, linear, nonhomogeneous equation.

b) T F The integrating factor used to solve the first order linear equation \( t^2 y' - 4ty = \sin 3t \) is \( \mu(t) = \frac{1}{t^4} \).

c) T F The equation \( \frac{dy}{dx} = \frac{e^{x+y}}{x^2 + y^2} \) is a separable equation.

d) T F The autonomous equation \( y' = y^2 \) has a semistable equilibrium solution.

e) T F If \( y_1 \) and \( y_2 \) form a set of fundamental solutions of the second order linear equation \( ay'' + by' + cy = 0, a \neq 0 \), then their Wronskian \( W(y_1, y_2) = 0 \).

f) T F The second order linear equation \( y'' + 4y = 0 \) has a general solution \( y(t) = C_1 \cos 2t + C_2 \sin 2t \).

g) T F For any oscillating mass-spring system undergoing free vibration, its natural frequency (when \( \gamma = 0 \)) is always greater than its quasi-frequency (when \( \gamma \neq 0 \)).

h) T F The inverse Laplace transform of \( F(s) = \frac{3}{(s - 2)^2} \) is \( f(t) = t^3 e^{2t} \).
14. (18 pts.) Use the Laplace transform to solve the following initial value problem.
\[ y'' + 4y' + 3y = \delta(t - 3) \quad y(0) = 0, \quad y'(0) = 2. \]

No credit will be given if the Laplace transform is not used to solve this problem.
15. (18 pts.)
   
   a) (15 pts.) Solve the initial value problem
   
   \[ x' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \]

   b) (3 pts.) Classify the type and stability of the critical point at the origin.
16. (18 pts.) Consider the nonlinear system:
\[ \begin{align*}
x' &= x - y \\
y' &= xy - 2x
\end{align*} \]

a) (4 pts.) Find all its critical points (there are two).

b) (4 pts.) Write down the matrix of partial derivatives used to linearize the system.

c) (10 pts.) Choose either one of the critical points you found in a), linearize the system about the critical point, then determine its type and stability by examining the linearized system.