

Math 251 – Sections 1/2
December 14, 2005 Final Exam

Name _____

Please check one of the boxes below

Section 1 – 1st per – 8:00am

Section 2 – 3rd per – 10:10am

There are 10 questions on this exam. Question 2, which has 12 parts, is worth 24 points. The other questions are worth 14 points each. The total number of points is 150. If a question has multiple parts, then the points assigned to the question are divided equally among the parts, unless otherwise indicated.

Where appropriate, **show your work** to receive credit; partial credit may be given.

Please turn your cell phone **OFF**.

The use of calculators, books, or notes is not permitted on this exam.

Time limit 1 hour and 50 minutes.

1. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 3 & \text{if } 0 \leq x < 1 \\ 0 & \text{if } 1 \leq x \end{cases}$$

Write down the first seven terms in the Fourier series of $f(x)$ on $[-2, 2]$.

2. In Parts **a.** through **e.** $s_n(x)$ denotes the n th partial sum of the Fourier series in Problem 1.

a. Find $\lim_{n \rightarrow \infty} s_n(4)$

b. Find $\lim_{n \rightarrow \infty} s_n(5)$

c. Find $\lim_{n \rightarrow \infty} s_n(6)$

d. Is it true that for all sufficiently large n : $s_n(x) \geq -0.15$ for every x in $[1.4, 1.7]$?

e. Is it true that for all sufficiently large n : $s_n(x) \leq 3.15$ for every x in $[0.9, 1.1]$?

f. What are the values of the following integrals?

$$\int_{-6}^6 \cos\left(\frac{\pi}{3}x\right) \cos\left(\frac{5\pi}{6}x\right) dx$$

$$\int_{-6}^6 \sin\left(\frac{\pi}{6}x\right) \sin\left(\frac{\pi}{6}x\right) dx$$

g. What are the values of the following integrals?

$$\int_0^6 \cos\left(\frac{\pi}{6}x\right) dx$$

$$\int_{-6}^6 \sin\left(\frac{\pi}{17}x\right) dx$$

h. If $u(x, t)$ is the temperature of a thin rod of length L insulated on its sides and ends, then what are $u_x(0, t)$ and $u_x(L, t)$ for any t ?

i. We can find a sine series for the function $f(x) = x$ on the interval $[0, 2]$. To what value does the sine series converge at $x = 2$?

j. We can also find a cosine series for the function $f(x) = x$ on the interval $[0, 2]$. To what value does the cosine series converge at $x = 2$?

k. One of the following two questions is much easier than the other. Answer the easier one.

I. What is the sine series for $f(x) = 7$ on $[0, 4]$?

II. What is the cosine series for $f(x) = 7$ on $[0, 4]$?

l. Suppose that a car does not pass the safety inspection because it crosses its equilibrium position more than once after being shaken up and down. Also suppose that there is a severe shortage of shock absorbers but an abundance of springs in the auto parts stores. So installing new shocks is not an option but would exchanging the current set of springs with ones that have a larger spring constant enable the car to pass the inspection?

3. Consider the two point boundary value problem

$$y'' + \lambda y = 0 \quad y'(0) = 0, \quad y'(2) = 0$$

You may assume that there are no eigenvalues with $\lambda < 0$.

- a. **2pt** Write down the general solution of the above ODE when $\lambda = 0$

- b. **2pt** Using your answer to Part a. determine whether or not $\lambda = 0$ is an eigenvalue. If it is, then what is the corresponding eigenfunction?

- c. **2pt** Write down the general solution of the above ODE when $\lambda > 0$

- d. **6pt** Using your answer to Part c. determine all eigenvalues and eigenfunctions with $\lambda > 0$.

- e. **2pt** Which one of the following partial differential equations can be solved by using the technique of separation of variables? Circle it:

$$u_t = u_x + 1 \quad u_t = u_x + u$$

4. Suppose a thin homogeneous rod 5 cm long is insulated along its sides and made of a material with thermal diffusivity $\alpha^2 = 0.8$ and that the left end is held at 10° and the right end is held at 60° ,

a. 4pt Find the initial temperature distribution that leads to a steady state solution to the above problem (ie, a solution that is constant with respect to t).

b. 6pt Consider the same problems as in Part **a**. EXCEPT that the initial temperature distribution is

$$f(x) = 10$$

Find the temperature $u(x, t)$ of the rod at any time $t > 0$ and at any point x inside the rod $0 < x < 5$. (If the answer involves finding a sine or cosine series then **DO NOT** find the actual values of the a_n and/or b_n which appear in the answer.)

c. 4pt Consider the same problem as in Part **b**.. After a long time passes, approximately what is the temperature of the rod at $x = 3$ cm?

5. **a. 4pt** Suppose a thin homogeneous rod 5 cm long is insulated along its sides and made of a material with thermal diffusivity $\alpha^2 = 0.8$. Assume that also the **ends** of this rod **are insulated**. and that the initial temperature distribution of the rod is

$$g(x) = x^3$$

Find the limit of the temperature $u(x, t)$ as $t \rightarrow \infty$.

- b. 4pt** The displacement $u(x, t)$ of a string of length 5cm with ends clamped satisfies the differential equation $4u_{xx} = u_{tt}$. If the initial displacement of the string is 0 and the initial velocity is given by $\sin(2\pi x)$, then what is the displacement $u(x, t)$ of the string at $t > 0$.

- c. 6pt** Now assume that the initial displacement of the string in Part **b** is $\sin(5\pi x)$ and the initial velocity of the string is 0. Write down a formula for the displacement of the the displacement $u(x, t)$ of the string at $t > 0$.

6. a. 4pt Find the solution of the Laplace equation on the rectangle

$$\{(x, y) \mid 0 < x < 5, \quad 0 < y < 7\}$$

which has the following values on the boundary:

$$u(0, y) = 0 \text{ if } 0 < y < 7$$

$$u(x, 0) = 0 \text{ if } 0 < x < 5$$

$$u(x, 7) = 0 \text{ if } 0 < x < 5$$

$$u(5, y) = 4 \sin(3\pi y) \text{ if } 0 < y < 7$$

- b. 4pt Consider the function $F(\theta) = 11 + 10 \cos(9\theta) + 8 \sin(7\theta)$ defined on $[-\pi, \pi]$ Find the solution to the Dirichlet problem for the unit disk with the values on the boundary given by $F(\theta)$. That is, find a function in polar coordinates $u(r, \theta)$, which is harmonic when $r < 1$ and which satisfies the following when $r = 1$:

$$u(1, \theta) = F(\theta) \quad \text{for} \quad -\pi < \theta < \pi$$

- c. 4pt If the function $F(\theta)$ in Part b. is replaced by $F(\theta) = |\theta|$ then find the value of the solution to the Dirichlet problem at the center of the disk: $u(0, 0)$.

- d. 2pt Now suppose that only thing known about the p.w. continuous function $F(\theta)$ in Part b. is that its values are between 10° and 20° . If $u(0, 0) = 20$, then what is $u(1/2, \pi/2)$?

7. a. Solve the following IVP. (**DO NOT** use Laplace transforms.)

$$y'' - 4y' + 4y = 0 \quad y(1.23) = 1, \quad y'(1.23) = -1$$

- b. Find a particular solution to the following nonhomogeneous linear ODE (using any method covered in Math 251):

$$y'' - 4y' + 4y = e^{2t}$$

8. In each part of this Problem do the following:
- Sketch a phase portrait for this system.
 - State the name associated with the critical point at $(0,0)$ and state whether it is stable, asymptotically stable or unstable?

a. **5pt** The homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ whose general solution is:

$$\mathbf{x} = c_1 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

b. **5pt** The homogeneous linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$.

c. **4pt** The homogeneous linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$.

9. Solve the following initial value problems:

a. $ty' = 2t^2e^{2t} + y \quad y(0) = 1$

b. $3t^2y' = y^4 \quad y(1) = 0$

10. a. **6pt** An object with mass 2 kg stretches a spring 4 m to equilibrium. At time $t = 0$ it is released 2 meters below its equilibrium position with an upward velocity of 3 meters/sec . At time $t = 6$ it is struck with a hammer and as a result its momentum is **decreased** by 7 kg-m/sec at that moment in time. At time $t = 8$ a constant external force of 9 Newtons is added. Write down a differential equation with initial conditions for the displacement $y(t)$ of the object below its equilibrium position. **DO NOT SOLVE THIS EQUATION**

- b. **8pt** Solve the following IVP:

$$y'' + 9y = \delta(t - 2) + u(t - 1) \quad y(0) = -1, \quad y'(0) = 1$$