

Math 251 – Sections 1/2
October 12, 2005 First Exam

There are 9 questions on this exam. Question 1 is worth 20 points. Questions 2 through 9 are worth 10 points each. The total number of points is 100. If a question has multiple parts, then the points assigned to the question are divided equally among the parts, unless otherwise indicated.

Show all your work. Partial credit may be given.

The use of calculators, books, or notes is not permitted on this exam.

Please turn off your cell phone before starting this exam.

Time limit 1 hour and 15 minutes.

SOLUTIONS

1. a. Suppose $y(t)$ is a solves the ODE: $y' = e^y$. What ODE does $y(t+1)$ solve?

ANS. This is an autonomous equation. So a solution shifted satisfies the SAME ODE.

For the initial value problems in parts **b.** and **c.** state whether or not a unique solution is guaranteed to exist. If the answer is yes and if it is possible to determine the largest interval on which the solution exists **without actually solving the equation**, then do so.

b. _____ $(t-1)y' - (y+1)^{2/3} = 0, \quad y(-2) = -1$

ANS. NO GUARANTEE because the partial derivative with respect to y fails to be continuous at $(-2, -1)$.

c. _____ $(t+t^2)y' + y = 0, \quad y(-2) = 0$

ANS. This is a linear equation with coeffs being discontinuous at 0 and -1 . So a unique solution exists on $I = (-\infty, -1)$.

In parts **d.** and **e.** assume that $L[y] = y'' + py' + qy$ and that $p(t)$ and $q(t)$ are continuous functions of t on the entire real axis $(-\infty, \infty)$.

d. One of the following functions **CANNOT** be a solution of $L[y] = 0$ on $(-\infty, \infty)$. Circle it:

0 1 t $t - \sin 2t$ $2t - \sin 2t$ $2 - \sin 2t$

ANS. Zero is a solution. The function $2t - \sin 2t$ has the same initial conditions at 0. It **CANNOT** be a solution.

e. If $L[y_1] = 0$, and $L[y_2] = t$, then one of the following solves: $L[y] = 2t$. Circle it.

$y_1 + y_2$ $2y_1$ $3y_1 - y_2$ $4y_1 + 2y_2$

ANS. By the superposition principle $L[4y_1 + 2y_2] = 2t$.

In parts **f.** and **g.** consider each differential equation on the left and match it with the statement on the right that best describes the long time behavior of a **NONZERO** solution

iii. _____ **f.** $y'' - 4y' + 5y = 0$ **i.** approaches either ∞ or $-\infty$ as $t \rightarrow \infty$

ii. oscillates with decaying amplitude as $t \rightarrow \infty$

iii. oscillates with increasing amplitude as $t \rightarrow \infty$

iv. _____ **g.** $y'' + 4y' + 4y = 0$ **iv.** approaches zero as $t \rightarrow \infty$

In parts **h.** through **j.** match the descriptions on the right with the ODE's on the left.

ii. _____ **h.** $y' = y + e^t$ **ii.** linear but not separable

iii. _____ **i.** $y' = \sqrt{ty}$ **iii.** separable but not linear

i. _____ **j.** $y' = t - ty$ **iv.** not separable and not linear

i. linear and separable

2. Consider the autonomous differential equation

$$y' = (y - 6)(2 - y) = 8y - 12 - y^2$$

a. **3pts** Find a formula for y'' in terms of y . (Remember that ' indicates derivative with respect to the independent variable t .)

ANS. $y'' = (8 - 2y)y' = (y - 6)(2 - y)$ by the chain rule.

b. **3pts** Determine the signs of y'' as y varies from $-\infty$ to ∞ and tabulate your results in the following table.

interval on y -axis	sign of y'	sign of y''	$-2y + 8$	$6 - y$	$y - 2$
$(-\infty, 2)$	+	-	+	-	-
$(2, 4)$	-	+	+	-	+
$(4, 6)$	-	-	-	-	+
$(6, \infty)$	+	+	-	+	+

c. **2pts** What are the equilibrium solutions and which is (are) **asymptotically stable** and which is (are) **unstable**?

ANS. $y = 6$ is **asymptotically stable** and $y = 2$ is **unstable**. $y = 4$ is not an equilibrium solution.

d. **2pts** Sketch a graph of the solution with $y(0) = 3$ indicating its concavity as accurately as possible.

ANS. Click here for Dirfield applet.

3. a. 5pts Find the general solution to the differential equation

$$2y - t^3 \cos t - ty' = 0 \quad t > 0$$

ANS. This is a linear equation: $y'' - (2/t)y = -t^2 \cos t$. We find $\mu = 1/t^2$. Therefore $y/t^2 = -\sin t + C$.

b. 3pts Find the general solution to the following ODE

$$ty^2 - y' = 0 \quad t > 0$$

ANS. This is a separable equation: $y'/y^2 = t$. So $-1/y = t^2/2 + C$. Also note that $y = 0$ solves the original equation and this was lost when dividing.

c. 2pts Find the solution to the differential equation in part **b.** which satisfies the initial condition $y(1) = 0$.

ANS. The lost solution which we recovered satisfies the IVP.

4. **a. 2pts** Suppose that the complex function $y = 5e^{(3+4i)t} + 5e^{(3-4i)t}$ solves a second order homogeneous linear equation. Find the imaginary part of y .

ANS. Note that $5e^{(3+4i)t}$ and $5e^{(3-4i)t}$ are complex conjugates of each other because *cosine* is an even function and *sine* is an odd function. Therefore when added they give a complex number with 0 as the imaginary part.

- b. 8pts** Find the solution to the following differential equation with initial conditions:

$$y'' - 2y' + 5y = 0 \quad y(-\sqrt{3}) = 1, \quad y'(-\sqrt{3}) = 5$$

ANS. The characteristic polynomial has roots $1 \pm 2i$. Therefore the general solution is $y = e^t(c_1 \cos 2t + c_2 \sin 2t)$. Also $y' = y + 2e^t(-c_1 \sin 2t + c_2 \cos 2t)$. We first solve the IVP $y(0) = 1$, $y'(0) = 5$ and find that $y_{\text{tmp}}(t) = e^t(\cos 2t + 2 \sin 2t)$. Therefore the answer is $y(t) = y_{\text{tmp}}(t + \sqrt{3})$.

5. Solve the following second order ODE with initial conditions:

$$y'' - e^{-y}(y')^3 = 0 \quad y(2) = 0 \quad y'(2) = 1/2$$

ANS. Note that $y = \text{const}$ does not solve this IVP because of the requirement $y'(2) = 1/2$; so we may lose $y = \text{const}$ in the solution process without losing points. This is the case of the missing t , so we view y as the indep variable temporarily. Set $v = y'$ and note that $y'' = v(dv/dy)$. The ODE is now: $v(dv/dy) = e^{-y}v^3$. This is a separable equation: $(dv/dy)/v^2 = e^{-y}$. We integrate both sides with respect to y : $-1/v = -e^{-y} + C$. Set $y = 0$ and $v = 2$ to find $C = -1$.

We now let t again be the independent variable: $1/y' = e^{-y} + 1$ This is again an easy separable ODE: $(e^{-y} + 1) = 1$. The solution is given by the equation $-e^{-y} + y = t + D$ Setting $t = 2$ and $y = 0$ gives that $D = -3$.

6. a. 2pt Which one of the following equations has a solution of the form $F(t, y) = C$? Circle it:

$$3t + 2y + (3t + 2y)y' = 0 \qquad 3t + 2y + (3y + 2t)y' = 0$$

ANS. For the second equation we find $F_{ty} = 2$ and $F_{yt} = 2$. So it is exact.

- b. 8pt Find the solution to the following exact differential equation which satisfies $y(1) = 0$:

$$(4y^3 + 2ty)y' + t + y^2 = 0$$

ANS. This is an exact differential equation:

$$t + y^2 + (4y^3 + 2ty)y' = 0$$

with $F_t = t + y^2$ and $F_y = 4y^3 + 2ty$.

Therefore $F = \frac{1}{2}t^2 + ty^2 + h(y)$

Now differentiate F with respect to y :

$F_y = 2ty + dh/dy = 4y^3 + 2ty$. So $h(y) = y^4$. Therefore

$$F(t, y) = \frac{1}{2}t^2 + ty^2 + y^4 = C$$

is the general solution to the equation. Setting $t = 1$ and $y = 0$ gives $C = \frac{1}{2}$.

7. A tank with a capacity of 250 liters initially contains a mixture of 300 grams of salt dissolved in 50 liters of water. Well stirred mixture leaves the tank at the rate of 3 liters/min and salt water mixture with a concentration of 2 grams/liter enters the tank at the rate of 6 liters/min.

a. **8pts** Let $Q(t)$ be the quantity of salt in the tank at time $t \geq 0$. Write down a differential equation and an initial condition for the quantity $Q(t)$ of salt in the tank at any time $t \geq 0$. (**Do not solve it.**)

ANS. The volume of mixture at any time t is given by $V(t) = 250 + 3t$ Therefore

$$Q' = (2)(6) - \frac{Q}{50 + 3t}3 \quad Q(0) = 300$$

b. **2pts** Without solving the above ODE, determine $Q'(0)$.

ANS. $Q'(0) = 12 - 300(3)/50 = -6$ grams/min.

8. Suppose that the MSRI (Microbe Spectrum Research Institute) discovers a new type of bacteria. The investigation of the bacteria growth rate gives the following result: If $B(t)$ denotes the number of bacteria at time t **days**, then

$$B' = 2B$$

If the most effective antibiotic available at MSRI is used to control the growth of this bacteria, then the rate of growth is reduced by 4×10^6 **bacteria per day**.

- a. **3pts** Write down a differential equation that describes $B'(t)$ when this antibiotic is used.

ANS.

$$B' = 2B - 4 \times 10^6$$

- b. **3pts** Sketch a direction field for this differential equation.

ANS. Click here for Dirfield applet.

- c. **2pts** Determine the largest number of bacteria which when present at $t = 0$ and can eventually be eliminated by using this antibiotic?

- c. **2pts** Determine the largest number of bacteria which when present at $t = 0$ and can eventually be eliminated by using this antibiotic?

ANS. Anything that is below the equilibrium solution: $y = 2 \times 10^6$, e.g. 1999999 bacteria.

- d. **2pts** If initially there are 6×10^6 million bacteria, then EXPLAIN which takes longer:

i. To increase the number of bacteria from 7×10^6 to 8×10^6

ii. To increase the number of bacteria from 8×10^6 to 9×10^6 ?

ANS. Above the equilibrium solution, B' becomes larger as B becomes larger. Therefore it takes less longer **i.** .

9. Consider the following second order linear ODE $t^2y'' - 2y = 0$. It has a solution $y_1 = \frac{1}{t}$. Find another solution y_2 of this ODE that is not a constant multiple of this solution.

ANS. **ANS.** We seek a function v of t such that $y_2 = vy_1 = v/t$ solves the given equation. So we plug y_2 into the left hand side:

$$\begin{aligned} -2(y_2 &= vt) \\ 0(y_2' &= v'/t - v/t^2) \\ t^2(y_2'' &= v''/t - v'/t^2 - v'/t^2 + 2v/t^3) \\ 0 &= v''/t - 2v'/t^2 \end{aligned}$$

This leads to the following ODE for the function v'

$$\frac{v''}{v'} = \frac{2}{t}$$

This leads to $\ln v' = 2 \ln t$. Hence $v' = t^2$. Finally $v = t^3/3$ and $y_2 = t^2/3$. The general solution is

$$c_1/t + c_2t^2$$