

ANSWER KEY

1. Consider the following nonlinear system of ODE's:
$$\begin{aligned} x' &= x - xy \\ y' &= xy - 2y \end{aligned}$$

a. 2pt Determine its critical points.

ANS. Factoring gives

$$\begin{aligned} 0 &= x(y - 1) \\ 0 &= y(2 - x) \end{aligned}$$

gives two critical points $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{x}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

b. 10pt Approximate this nonlinear system near each one of its critical points by a linear system of ODE's. State the type and stability of the critical point of approximating linear system.

ANS. At $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the linearization is
$$\begin{aligned} x' &= -x \\ y' &= 2y \end{aligned}$$

which has eigenvalues $r_1 = -1$ and $r_2 = 2$, opposite signs.

The corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Hence the origin is a saddle for for this system, which is unstable, and the system has trajectories along the x -axis moving towards the origin and trajectories along the y -axis moving away the origin.

At $\mathbf{x}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ we set
$$\begin{aligned} u &= x - 2 \\ v &= y - 1 \end{aligned}$$

Substituting this into the original equation gives

$$\begin{aligned} u' &= (u + 2)v \approx 2v \\ v' &= (v + 1)(-u) \approx -u \end{aligned}$$

The eigenvalues of this system $\pm\sqrt{2}$ are purely imaginary. Thus this critical point is a center for the approximating linear system. The elliptical trajectories are traced in the clockwise direction.

c. 3pt Sketch a phase portrait near each of the critical points.

(2 pt for finding each the critical pt).

At $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: (2 pt for linearization at origin)

(2 pt for the eigenvalue eigenvector pairs)
(2 pt for the words "Saddle" and "Unstable")

At $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (1 pt for definition of u and v)

(4 pt for linearization at the second critical point)

(2 pt for the words "Center" and "Stable")

(1pt for sketching saddle at origin)

(1pt for sketching ellipse at $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$)

(1pt for getting both orientations.)

2. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } -2 \leq x < 0 \\ 3 & \text{if } 0 \leq x < 1 \\ 0 & \text{if } 1 \leq x \leq 2 \end{cases}$$

a. 12pt Find the Fourier series of $f(x)$ on $[-2, 2]$. (You may either write the first seven terms or use summation notation to write the entire series.)

ANS.

$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-2}^2 f(x) dx = \frac{1}{2} \int_0^1 3 dx = \frac{3}{2} \\ a_n &= \frac{1}{2} \int_{-2}^2 f(x) \cos\left(\frac{n\pi}{2}x\right) dx \\ &= \frac{3}{2} \int_0^1 \cos\left(\frac{n\pi}{2}x\right) dx \\ &= \frac{3}{2} \frac{2}{n\pi} \left[\sin\left(\frac{n\pi}{2}x\right) \right]_0^1 \\ &= \frac{3}{n\pi} \sin\left(\frac{3n\pi}{2}\right) \\ b_n &= \frac{1}{2} \int_{-2}^2 f(x) \sin\left(\frac{n\pi}{2}x\right) dx \\ &= \frac{3}{2} \int_0^1 \sin\left(\frac{n\pi}{2}x\right) dx \\ &= \frac{-3}{2} \frac{2}{n\pi} \left[\cos\left(\frac{n\pi}{2}x\right) \right]_0^1 \\ &= \frac{1}{n\pi} \left(1 - \cos\left(\frac{n\pi}{2}\right) \right) \end{aligned}$$

$$\frac{3}{4} + \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin\left(\frac{3n\pi}{2}\right) \cos\left(\frac{n\pi}{2}x\right) + \frac{1}{n\pi} \left(1 - \cos\left(\frac{n\pi}{2}\right) \right) \sin\left(\frac{n\pi}{2}x\right)$$

(1 pt for finding a_0)

(4 pt for finding a_n)

(4 pt for finding b_n)

(1 pt for placing $a_0/2$ in the final answer.)

(2 pt for placing a_n, b_n in front of $\cos\left(\frac{n\pi}{2}x\right)$. and $\sin\left(\frac{n\pi}{2}x\right)$.) in final answer)

b. 3pt Find $\lim_{n \rightarrow \infty} s_n(9)$ where $\{s_n(x)\}$ denotes the sequence of partial sums for the Fourier series found in Part **a**.

ANS. $\frac{3}{2}$.

(3 pt, all or nothing)

3. **a. 12pt** Find the sine series of the function $f(x) = x$ on the interval $[0, 3]$ (You may either write the first four terms or use summation notation to write the entire series.)

ANS.

$$\begin{aligned} b_n &= \frac{1}{3} \int_{-3}^3 x \sin\left(\frac{n\pi}{3}x\right) dx \\ &= \frac{2}{3} \int_0^3 x \sin\left(\frac{n\pi}{3}x\right) dx \\ &= \left[\frac{-2}{n\pi} x \cos\left(\frac{n\pi}{3}x\right) + \frac{6}{\pi^2 n^2} \sin\left(\frac{n\pi}{3}x\right) \right]_0^3 \\ &= \frac{-6}{n\pi} \cos(n\pi) \end{aligned}$$

Thus the sine series is

$$\sum_{n=1}^{\infty} \frac{-6}{n\pi} \cos(n\pi) \sin\left(\frac{n\pi}{3}x\right)$$

(2 pt for correct formula b_n .)

(8 pt for correctly evaluating the integrals)

(2 pt for correctly placing the coefficients in a sine series)

- b. 3pt** Find $\lim_{n \rightarrow \infty} s_n(6)$ where $\{s_n(x)\}$ denotes the sequence of partial sums for the sine series found in Part **a**.

ANS. 0

(3 pt, all or nothing)

4. 15pt Consider the two point boundary value problem

$$y'' + \lambda y = 0 \quad y'(0) = 0, \quad y(3) = 0$$

Find ALL eigenvalues and the corresponding eigenfunctions. Please check whether there are any eigenvalues for $\lambda < 0$, for $\lambda = 0$, for $\lambda > 0$. **Show your work**

ANS. Consider three cases: $\lambda = 0, \lambda < 0, \lambda > 0$

If $\lambda = 0$, the general solution is $y = c_1x + c_2$ and $y' = c_1$. Set $x = 0$ and see that $c_2 = 0$. Set $x = 3$ and see that $c_1 = 0$. $\lambda = 0$ is not an eigenvalue.

(1 pts for general solution.)

(2 pts for calculation of c_1 and c_2 .)

If $\lambda < 0$, the general solution is

$$y = c_1 \cosh(\sigma x) + c_2 \sinh(\sigma x)$$

where $\sigma = \sqrt{-\lambda}$. We will also need y'

$$y' = \sigma(c_1 \sinh(\sigma x) + c_2 \cosh(\sigma x))$$

Setting $x = 0$ in $y'(x)$ gives $0 = c_2 \cosh(0)$. So $c_2 = 0$. Set $x = 3$ in $y(x)$ and we see that $c_1 \cosh(3\sigma) = 0$ and see that c_1 must also be 0. There are no eigenvalues with $\lambda < 0$.

(2 pts for general solution.)

(3pts for the calculation of c_1, c_2)

If $\lambda > 0$, the general solution is

$$y = c_1 \cos(\sigma x) + c_2 \sin(\sigma x)$$

where $\sigma = \sqrt{\lambda}$. Also

$$y' = \sigma(-c_1 \sin(\sigma x) + c_2 \cos(\sigma x))$$

Setting $x = 0$ in $y'(x)$ gives $0 = c_2 \cos(0)$. So $c_2 = 0$. We now set $x = 3$ in $y(x)$ and we see that $c_1 \cos(3\sigma) = 0$. In order to avoid having also $c_1 = 0$ we need to have $3\sigma = (n + 1/2)\pi$, n any nonnegative integer. Thus we obtain the following eigenvalues and eigenfunctions:

$$\lambda = \left(\frac{(n + 1/2)\pi}{3} \right)^2 \quad \cos \left(\frac{(n + 1/2)\pi}{3} x \right)$$

(2 pts for general solution.)

(1pt for the calculation of c_2)

(2pt for the value of σ and/or λ that enables $c_1 \neq 0$)

(2pt for the correct eigenfunction)

5. **a. 12pt** Suppose a thin rod 5 cm long is insulated along its sides and made of a homogeneous material with thermal diffusivity $\alpha^2 = 0.8 \text{ cm}^2/\text{sec}$. Also suppose that the left end is held at constant temperature of 10° and the right end is held at constant temperature of 20° . Assume that the initial temperature distribution is

$$f(x) = 10$$

Find the temperature $u(x, t)$ of the rod at any time $t > 0$ and at any point x inside the rod $0 < x < 5$. In this problem **DO NOT EVALUATE** any integrals, however, make sure you indicate clearly what the integrands in the integrals are.

ANS. Here we break $f(x)$ into the sum of two pieces $g(x) = 10 + 2x$ and

$$h(x) = -2x$$

Along with $g(x)$ we keep the ends of the rod at constant temperature 10° and 20° and with $h(x)$ the ends of the rod are kept in ice water. So $u_{\text{steady}}(x, t) = g(x)$ solves the first part of the problem. To solve the second part we need to find a sine series for $h(x)$ on $[0, 5]$. We find the b_n with by evaluating the following integrals

$$b_n = \frac{1}{5} \int_{-5}^5 -2x \sin\left(\frac{n\pi}{5}x\right) dx$$

We can now write the solution to the second problem:

$$u_{\text{transient}}(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{5}x\right) e^{-.8(n\pi/5)^2 t}$$

And finally,

$$u(x, t) = u_{\text{steady}}(x, t) + u_{\text{transient}}(x, t)$$

(3 pt for correctly breaking the problem into a sum of two.)

(3 pt for the steady state solution.)

(3 pt for stating a sine series is needed and correct integrand for b_n).

(3 pt for correctly placing an exponential function next to the sines in the sine series.)

b. 3pt Find $\lim_{t \rightarrow \infty} u(2, t)$

ANS. The steady state solution evaluated at $x = 2$: $10 + 2(2) = 14$.

(3 pts for evaluating whatever is given as the steady state solution in Part a at $x = 2$ and no credit for anything else.)

6. Suppose a thin homogeneous rod 4 cm long is insulated along its sides and made of a material with thermal diffusivity = $0.8\text{cm}^2/\text{sec}$. Also assume that the **ends** of this rod **are insulated**.

a. **4pt** If the initial temperature distribution of the rod is

$$g(x) = x^3$$

then find the limit of the temperature $u(x, t)$ as $t \rightarrow \infty$.

ANS. After a long time the temperature at any point of an insulated rod approaches the initial average temperature which does not change.

$$\frac{a_0}{2} = \frac{1}{4} \int_0^4 x^3 dx = \frac{4^4}{4^2} = 16$$

(2pt for stating its the average temp or $a_0/2$.)
(2pt for the correct value.)

b. **7pt** If the initial temperature distribution of the rod is

$$g(x) = \cos\left(\frac{\pi}{4}x\right)$$

then what is the temperature of the rod at x -units from the left end at time $t > 0$?

ANS.

$$\cos\left(\frac{\pi}{4}x\right) e^{-.8(\pi/4)^2t}$$

(7 pt for correctly putting the exponential next to the cosine).

c. **4pt** For the same initial temperature as in Part **b**. determine for which x the temperature is negative when $t = 0.15$.

ANS. Whatever makes $\cos\left(\frac{\pi}{4}x\right)$ That is, the second half of the rod: $2 < x < 4$.

(2 pt for stating that its cosine that needs to be looked at.)
(2 pt getting the interval correct.)

7. **a. 8pt** Consider the function $F(\theta) = 7\cos(8\theta) + 9\sin(10\theta) + 11$ defined on $[-\pi, \pi]$. Find the solution to the Dirichlet problem for the unit disk with the values on the boundary given by $F(\theta)$. That is, find a function in polar coordinates $u(r, \theta)$, which is harmonic when $r < 1$ and which satisfies the following when $r = 1$:

$$u(1, \theta) = F(\theta) \quad \text{for} \quad -\pi < \theta < \pi$$

ANS. $F(\theta) = 7r^8 \cos(8\theta) + 9r^{10} \sin(10\theta) + 11$

(8pt for the correct answer.

It seems almost impossible to make an error on this one.)

- b. 7pt** If the function $F(\theta)$ in Part **a.** is replaced by $F(\theta) = \theta^2$ then find $u(0, 0)$, the value of the solution at the center of the disk.

ANS. The average temp on the boundary is the temp at the center:

$$a_0/2 = \frac{2}{\pi} \int_0^{\pi} \theta^2 d\theta = \frac{2}{3\pi} \pi^3 = \frac{2}{3} \pi^2$$

(3pt for stating that its the average temp.)

(4pt for finding the average.)

8. 7pt a. The displacement $u(x, t)$ of a string of length 5cm with ends clamped satisfies the differential equation $3u_{xx} = u_{tt}$. If the initial displacement of the string is 0 and the initial velocity is given by $\sin(2\pi x)$, then what is the displacement $u(x, t)$ of the string at $t > 0$.

ANS. The solution is $\frac{1}{\sqrt{32\pi}} \sin(2\pi x) \sin(\sqrt{3}2\pi t)$.

(3pt for writing a product of sines)

(2pt for getting $\sqrt{3}$ in the coeff of t)

(2pt for getting the coeff of in front correct)

b. 8pt Now assume that the initial displacement of the string in Part **a** is $\sin(5\pi x)$ and the initial velocity of the string is 0. Write down a formula for the displacement $u(x, t)$ of the string at $t > 0$.

ANS. The solution is $\sin(2\pi x) \cos(\sqrt{3}2\pi t)$. (4pt for writing a product of sine and cosine)

(4pt for getting $\sqrt{3}$ in the coeff of t)

9. 15pt A salt water mixture in a tank with capacity 500 L is constantly being mixed to keep it homogeneous. A mixture with concentration of 1 kg salt per liter water flows in at the rate of 3 L/min. The homogeneous mixture flows out the same rate of 2 L/min. Initially there are 300 kg dissolved in 200 L of water in the tank. Determine the amount of salt in the tank at the moment when the tank is filled to its capacity.

ANS. In this case the volume of mixture in the tank at any time t varies according to the following formula:

$$V(t) = 200 + 2t$$

and therefore the differential equation satisfied by $Q(t)$ is as follows:

$$Q' = 3 - \frac{Q}{200 + 2t} \quad Q(0) = 300$$

$$Q' + \frac{1}{2} \frac{Q}{100 + t} = 3 \quad Q(0) = 300$$

Now the integrating factor becomes $\exp(\frac{1}{2} \ln(100 + t)) = (100 + t)^{1/2}$ and we obtain:

$$((100 + t)^{1/2}Q)' = 3(100 + t)^{1/2}$$

By integration we obtain

$$(100 + t)^{1/2}Q = 2(100 + t)^{3/2} + C$$

and setting $t = 0$ gives

$$C = 100^{1/2}300 - 2(100)^{3/2} = 100^{3/2}$$

Therefore,

$$Q = 2(100 + t) + 100^{3/2}(100 + t)^{-1/2}$$

The tank fills when $t = 150$ and we find that

$$Q(150) = 500 + 100\left(\frac{2}{5}\right)^{1/2}\text{kgrams}$$

(1 pt for formula for volume)

(4 pt for the ODE for Q')

(1 pt for initial condition)

(2 pt for finding the integrating factor)

(3 pt for integrating the ODE)

(2 pt for determining constant in ODE)

(2 pt for determining the time of filling and plugging into formula for $Q(t)$)

10. **a. 12pt** Assume that acceleration due to gravity g is equal to 10 meter/sec². An object whose mass is 1 kg stretches a spring $\frac{10}{9}$ meters. The object is connected to a damper with damping constant $\gamma = 6$ Newton-sec/meter and an external force equal to $\cos 3t$ Newton is also applied. The object is pushed up 1 meter above its equilibrium position and then set into motion with a downward velocity of 3 meter/sec. Then determine the displacement $y(t)$ of the object below its equilibrium position at any $t > 0$.

ANS. Since $mg = kL$ and $g = 10, m = 1, L = 10/9$ we have $k = 9$, the differential equation for the displacement $y = y(t)$ is

$$y'' + 6y' + 9y = \cos 3t$$

The initial conditions satisfied by $y(t)$ are $y(0) = -1$ and $y'(0) = 3$. The characteristic polynomial has a double root $r = -3$. The general solution to the associated homogeneous ODE is $y = e^{-3t}(c_1t + c_2)$.

To find a solution to the nonhomogeneous ODE complexify:

$$y'' + 6y' + 9y = e^{3it}$$

and plug in $y_c = Ae^{3it}$ because $3i$ is not a root.

$$\begin{aligned} 9(y_p &= Ae^{3it}) \\ 6(y'_p &= A(3i)e^{3it}) \\ 1(y''_p &= A(-9))e^{3it} \end{aligned}$$

We see that $A18i = 1$ or $A = -i/18$. Taking the real part gives a particular solution: $y_p = (1/18)\sin 3t$. And consequently,

$$y = e^{-3t}(c_1t + c_2) + (1/18)\sin 3t$$

is the general solution. Also

$$y' = -3y + c_1e^{-3t} + (1/6)\cos 3t$$

We now solve the IVP by setting $t = 0$ in y and y' :

$$-1 = c_2$$

$$3 = 3 + c_1 + 1/6 \quad c_1 = -1/6$$

Finally,

$$y = e^{-3t}(-t/6 - 1) + (1/18)\sin 3t$$

(2 pts for the ODE)

(1 pts for initial conditions)

(3 pts for general solution to the homogeneous equation condition)

(4 pts for the particular solution the nonhomogeneous equation)

(2 pts for setting up the IVP)

- b. 3pt** What is the transient part of the solution to Part **a** and what is the steady state part?

ANS. Transient part: $e^{-3t}(-t/6 - 1)$ Steady State part: $(1/18)\sin 3t$

(2pts for correct transient part) (1pts for correct steady state part)