Math 251
December 14, 2005 Answer Key to Final Exam

Name __________________________
Section _____

There are 10 questions on this exam. Many of them have multiple parts. The point value of each question is indicated either at the beginning of each question or at the beginning of each part where there are multiple part. Where appropriate, show your work to receive credit; partial credit may be given.

The use of calculators, books, or notes is not permitted on this exam.

Please turn off your cell phone.

Time limit 1 hour and 50 minutes.

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1. Consider the nonlinear system:

\[ x' = 2x - xy \]
\[ y' = -3y + xy \]

a. 2pt Find all the critical points of the nonlinear system.

ANS. \[ 0 = 2x - xy = x(2 - y) \] and \[ 0 = -3y + xy = y(-3 + x) \]
gives two critical points \( x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) and \( x_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \).

Give 1pt for each correctly found critical pt.

b. 6pt In a neighborhood of each critical point approximate the nonlinear system by a linear system.

ANS. At \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) the linearization is \[ x' = 2x \]
\[ y' = -3y \]
which has eigenvalues \( r_1 = 2 \) and \( r_2 = -3 \)
which have opposite sign.

Give 2pt for linearizing correctly.

At \( x_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \) we set \( u = x - 3 \)
\( v = y - 2 \). Substituting this into the original equation gives

\[ u' = (u + 3)(-v) \approx -3v \]
\[ v' = (v + 2)u \approx 2u \]

The eigenvalues of this system are purely imaginary.

Give 2pt for substituting new variables correctly. Give 2pt for linearizing correctly.

c. 2pt Determine the name and the stability of the the critical points of each of the linear approximations.

At the origin the critical pt is a saddle which is always unstable. At \( \begin{pmatrix} 3 \\ 2 \end{pmatrix} \) the critical pt is a center which is always stable.

Give 1pt for naming the each critical pt and its stability correctly.

d. 2pt Sketch a phase portrait for the original nonlinear system.

ANS. To see the phase portrait enter the system into the [phase portrait applet.]
The orientation of the center can be determined by plugging (4, 2) into the original system.

Give 1pt for the phase portrait drawn correctly near each critical point, including orientation.

e. 2pt The linearization of a nonlinear system may have a critical point that is not guaranteed to reflect the behavior of the original nonlinear system. List the three types of critical points for which this may happen.

ANS. center, proper and improper node. Give 0pt for 1 correct.

Give 1pt for 2 correct.

Give 2pt for 3 correct.
2. Consider the function

\[ f(x) = \begin{cases} 
0 & \text{if } x < 0 \\
3 & \text{if } 0 \leq x < 1 \\
0 & \text{if } 1 \leq x
\end{cases} \]

a. 10pt Find the Fourier series of \( f(x) \) on \([-2, 2]\). (Either use summation notation to write the answer or write the first seven terms.)

ANS.

\[ a_0 = \frac{1}{2} \int_{-2}^{2} f(x) \, dx = \frac{1}{2} \int_{0}^{1} 3 \, dx = 3 \]

Give 2pt for correct \( a_0 \)

\[ a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos \left( \frac{n\pi x}{2} \right) \, dx \]

\[ = \frac{1}{2} \int_{0}^{1} 3 \cos \left( \frac{n\pi x}{2} \right) \, dx = \frac{3}{2} \frac{2}{n\pi} \left[ \sin \left( \frac{n\pi x}{2} \right) \right]_{0}^{1} \]

\[ = \frac{3}{n\pi} \left[ \sin \left( \frac{n\pi}{2} \right) \right] \]

Give 2pt for correct expression for \( a_n \) after substituting \( f(x) \) into the integral. Give 1pt for correct evaluation of integral.

\[ b_n = \frac{1}{2} \int_{-2}^{2} f(x) \sin \left( \frac{n\pi x}{2} \right) \, dx \]

\[ = \frac{1}{2} \int_{0}^{1} 3 \sin \left( \frac{n\pi x}{2} \right) \, dx = -\frac{3}{2} \frac{2}{n\pi} \left[ \cos \left( \frac{n\pi x}{2} \right) \right]_{0}^{1} \]

\[ = \frac{3}{n\pi} \left[ 1 - \cos \left( \frac{n\pi}{2} \right) \right] \]

Give 2pt for correct expression for \( b_n \) after substituting \( f(x) \) into the integral. Give 1pt for correct evaluation of integral.

The Fourier series is

\[ \frac{3}{2} + \sum_{n=1}^{\infty} \frac{3}{n\pi} \left[ \sin \left( \frac{n\pi}{2} \right) \right] \cos \left( \frac{n\pi x}{2} \right) + \frac{3}{n\pi} \left[ 1 - \cos \left( \frac{n\pi}{2} \right) \right] \sin \left( \frac{n\pi x}{2} \right) \]

Give 2pt for correct assembly of formulas for \( a_0, a_n, b_n \) into a Fourier series. Take off 1pt for not dividing \( a_0 \) by 2.

In Parts b. through d. \( s_n(x) \) denotes the partial sums of the Fourier series on \([-2, 2]\) of the function \( f(x) \).

b. 2pt Find \( \lim_{n \to \infty} s_n(7) \)

ANS. = \( \lim_{n \to \infty} s_n(-1) = 0 \)

Give 2pt and NO PARTIAL CREDIT.

c. 2pt Find \( \lim_{n \to \infty} s_n(8) \)

ANS. = \( \lim_{n \to \infty} s_n(0) = 3/2 \) Give 2pt and no partial credit.

d. 2pt Find \( \lim_{n \to \infty} s_n(8.5) \)

ANS. = \( \lim_{n \to \infty} s_n(1/2) = 3 \) Give 2pt and NO PARTIAL CREDIT.
3. a. 3pt Which of the following already has the form of a Fourier series on the interval \([-2, 2]\). Explain!

\[ f(x) = 4 \sin \left( \frac{\pi}{3} x \right) \quad g(x) = \frac{1}{4} + 2 \cos(3\pi x) \]

ANS. The second integral because \( \left( \frac{n\pi}{2} \right) = 3\pi \) when \( n \) is a positive integer but \( \left( \frac{n\pi}{2} \right) = \frac{\pi}{3} \) only when \( n = 2/3 \) which is not a positive integer.
Give 3pt and no partial credit. Give zero points if a statement saying that “\( n = 2/3 \) IS NOT AN INTEGER” does not appear.

b. 3pt We can find a sine series for the function \( f(x) = x^3 \) on the interval \([0, 2]\). To what value does the sine series converge at \( x = 2 \)? Explain!

ANS. The odd extension is \(-8\) at \( x = -2 \). The average of 8 and \(-8\) is 0.
Give 3pt and no partial credit.

c. 3pt We can also find a cosine series for the function \( f(x) = x^3 \) on the interval \([0, 2]\). To what value does the cosine series converge at \( x = 2 \)? Explain!

ANS. The even extension is 8 at \( x = -2 \). The average of 8 and 8 is 8.
Give 3pt and no partial credit.

d. 3pt Which one of the following partial differential equations can be solved by using the technique of separation of variables?

\[ u_t = u_x + 1 \quad u_t + u_x = u \]

Explain!

ANS. We try to plug in \( u = XT \) and separate variables.
In first case \( XT' = X'T + 1 \) which cannot be separated.
In the second case \( XT' = X'T + XT \) which separates very nicely
\[ \frac{T'}{T} = 1 - \frac{X'}{X} \]
Give 2pt for plugging in \( XT \) correctly into either PDE. Give 1pt for indicating how \( XT' = X'T + XT \) can be separated.
4. 14pt Consider the function

\[ g(x) = \begin{cases} 
  x & \text{if } 0 \leq x < 1 \\
  0 & \text{if } 1 \leq x \leq 3 
\end{cases} \]

Find a cosine series for the function \( g(x) \) on the interval \([0, 2]\). (Either use summation notation to write the answer or write the first four terms.)

\[ a_0 = \frac{1}{2} \int_{-2}^{2} g_e(x) \, dx = \int_{0}^{2} g(x) \, dx = \int_{0}^{1} x \, dx = \frac{1}{2} \]

Equivalently, \( a_0 \) is \( 1/2 \) the area below the graph of \( g_e \) which is the area of a square of side 1. Give 3pt and no partial credit.

\[ a_n = \frac{1}{2} \int_{-2}^{2} g_e(x) \cos \left( \frac{n\pi}{2} x \right) \, dx = \int_{0}^{2} g(x) \cos \left( \frac{n\pi}{2} x \right) \, dx = \int_{0}^{1} x \cos \left( \frac{n\pi}{2} x \right) \, dx = \frac{2}{n\pi} \left[ x \sin \left( \frac{n\pi}{2} x \right) \right]_{0}^{1} - \frac{2}{n\pi} \int_{0}^{1} \sin \left( \frac{n\pi}{2} x \right) \, dx = \frac{2}{n\pi} \sin \left( \frac{n\pi}{2} \right) + \frac{4}{n^2\pi^2} \left[ \cos \left( \frac{n\pi}{2} \right) - 1 \right] \]

Give 6pt for correct expression for \( a_n \) after substituting \( f(x) \) into the integral. Give 4pt for correct evaluation of integral.

The cosine series is:

\[ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \left( \frac{n\pi}{2} \right) + \frac{4}{n^2\pi^2} \left[ \cos \left( \frac{n\pi}{2} \right) - 1 \right] \cos \left( \frac{n\pi}{2} x \right) \]

Give 2pt for correct assembly of all ingredients into a cosine series.
5. 12pt Determine all POSITIVE eigenvalues \( \lambda \) and the corresponding eigenfunctions for the following two point boundary value problem:

\[
y'' + \lambda y = 0 \quad y'(0) = 0, \quad y(3) = 0
\]

**ANS.** The characteristic polynomial is \( r^2 + \lambda \) and it has purely imaginary roots since \( \lambda > 0 \). Set \( \sigma = \sqrt{\lambda} \). Then the roots are \( \pm \sigma i \).

In this case the general solution is

\[
y = c_1 \cos(\sigma x) + c_2 \sin(\sigma x)
\]

We will also need \( y' \) so let’s write it down here.

\[
y' = \sigma (-c_1 \sin(\sigma x) + c_2 \cos(\sigma x))
\]

Give 4pt for correct expression of \( y, y' \)

Setting \( x = 0 \) in \( y(x) \) gives \( 0 = c_1 \cos(0) + c_2 0 \). So \( c_1 = 0 \).

Give 2pt for eliminating \( c_1 \)

We now set \( x = 3 \) in \( y'(x) \) and we see that \( \sigma c_2 \cos(3\sigma) = 0 \). In order to avoid having also \( c_2 = 0 \) we need to have \( 3\sigma = (n + 1/2)\pi, \) n any nonnegative integer. Thus we obtain the following eigenvalues and eigenfunctions:

\[
\lambda = \left( \frac{(n + 1/2)\pi}{3} \right)^2
\]

Give 4pt for the eigenvalues

The eigenfunctions are

\[
\cos \left( \frac{(n + 1/2)\pi}{3} \right)
\]

Give 2pt for the eigenfunctions. Take off 1pt for the sin instead of cos.
6. Suppose a thin homogeneous rod 5 cm long is insulated along its sides and made of a material with thermal diffusivity $\alpha^2 = 0.8$ and that the left end is held at $10^\circ$ and the right end is held at $60^\circ$.

a. 2pt What is the steady state solution to the above problem?

**ANS.** It is a linear function which is 10 at $x = 0$ and 60 at $x = 5$ $10 + 10x$.

Give 2pt and no partial credit.

b. 8pt If the initial temperature of the above rod is $60^\circ$ then find the temperature $u(x, t)$ of the rod at any time $t > 0$ and at any point $x$ inside the rod $0 < x < 5$. (If the answer involves finding a sine or cosine series then **DO NOT** find the actual values of the $a_n$ and/or $b_n$ which appear in the answer but indicate clearly what integrals must be evaluated to find them.)

**ANS.** We decompose the problem into the sum of two problems: one which has a steady state solution and the other that has $0^\circ$ at the ends. The initial temperature $f(x)$ of the the ends in ice water problem when added to the steady state solution must be $60^\circ$. Therefore $f(x) = 50 − 10x$.

Give 4pt and no partial credit for identifying $f(x) = 50 − 10x$.

The solution is as follows

$$u(x, t) = 10 + 10x + \sum_{n=1}^{\infty} \left[ \frac{2}{5} \int_{0}^{5} f(x) \sin \left( \frac{n\pi x}{5} \right) \, dx \right] \sin \left( \frac{n\pi x}{5} \right)$$

Give 2pt for the correct integral for $a_n$. Give 2pt for the assembling all ingredients correctly.

c. 4pt For the same problem as in Part b., approximately what is the temperature of the rod at $x = 3$ cm, after a long time.

**ANS.** The steady state solution evaluated at $x = 3$: $10 + 10(3) = 40$.

Give 4pt and no partial credit.
7. a. 6pt The displacement \( u(x, t) \) of a string of length 5cm with ends clamped satisfies the differential equation \( 9u_{xx} = u_{tt} \). Suppose that the initial displacement of this string is \( \sin(5\pi x) \) and the initial velocity of the string is 0. Write down a formula for the displacement \( u(x, t) \) of the string at \( t > 0 \).

**Hint:** \( \sin(5\pi x) \) is already in the form of a Fourier series.

**ANS.**

The form of the solution is \( C \sin(5\pi x) \cos(qt) \) or \( C \sin(5\pi x) \sin(qt) \).

Give 2pt for identifying the form.

For zero initial velocity choose the first form with \( q = 2(5\pi) = 10\pi \) and \( C = 1 \). i.e., \( u(x, t) = \sin(5\pi x) \cos(10\pi t) \)

Give 2pt for identifying \( q \) and no partial credit. Give 2pt for identifying \( C \) and no partial credit.

b. 6pt Consider two identical thin rods having length 6cm and which are insulated except perhaps at their ends. Also suppose that their temperatures \( u(x, t) \) satisfy the following boundary conditions.

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<td>I</td>
<td>II</td>
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<tr>
<td>( u(x, 0) = 30 ) for ( 0 \leq x \leq 6 )</td>
<td>( u(x, 0) = 30 ) for ( 0 \leq x \leq 6 )</td>
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<tr>
<td>( u(0, t) = 0 = u(6, t) ) for ( t &gt; 0 )</td>
<td>( u_x(0, t) = 0 = u_x(6, t) ) for ( t &gt; 0 )</td>
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Determine which will be warmer after a long time. Explain.

**ANS.** I represents a rod with ends in ice water. II represents a rod which is completely insulated.

Give 1pt for each of these statements.

The long time temp of I is \( 0^\circ \)

Give 2pt for this statement and no partial credit.

The long time temp of II is the average temperature at \( t = 0 \) which is \( 30^\circ \).

Give 2pt for this statement and no partial credit.
8. a. 8pt Find the solution of the Laplace equation on the rectangle

\[ \{(x, y) \mid 0 < x < 5, \ 0 < y < 7\} \]

which has the following values on the boundary:
- \( u(0, y) = 0 \) if \( 0 < y < 7 \)
- \( u(x, 0) = 0 \) if \( 0 < x < 5 \)
- \( u(x, 7) = 0 \) if \( 0 < x < 5 \)
- \( u(5, y) = f(y) \) if \( 0 < y < 7 \)

(Note: Your answer requires a sine or cosine series for \( f(y) \). Since no formula for \( f(y) \) is given, write but do not try to evaluate the formula for \( a_n \) or \( b_n \).)

ANS. The form of the “basic building block” for this problem is
\[ C \sinh(px) \sin \left( \frac{n\pi}{7} y \right) \]

Give 2pt for this statement and no partial credit.

We need to have \( p = \frac{n\pi}{7} \) in order to satisfy Laplace’s eqn.

Give 2pt for this statement and no partial credit.

We need to choose \( C = \frac{1}{\sinh \left( \frac{n\pi}{7} 5 \right)} \) in order to correct the boundary values on the right edge of the rectangle.

Therefore each term of a solution to this problem must have the form:
\[ \frac{1}{\sinh \left( \frac{n\pi}{7} 5 \right)} \sinh \left( \frac{n\pi}{7} x \right) \sin \left( \frac{n\pi}{7} y \right) \]

Give 2pt for this statement and no partial credit.

Finally if we have a sine series \( f(y) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi}{7} y \right) \), then the solution is:
\[ \sum_{n=1}^{\infty} \frac{b_n}{\sinh \left( \frac{n\pi}{7} 5 \right)} \sinh \left( \frac{n\pi}{7} x \right) \sin \left( \frac{n\pi}{7} y \right) \]

Give 2pt for this statement and no partial credit.

b. 8pt Consider the function \( F(\theta) = 11 + 10 \cos(9\theta) + 8 \sin(7\theta) \) defined on \([−\pi, \pi]\) Find the solution to the Dirichlet problem for the unit disk with the values on the boundary given by \( F(\theta) \). That is, find a function in polar coordinates \( u(r, \theta) \), which is a solution of Laplace’s equation when \( r < 1 \) and which satisfies the following when \( r = 1 \):

\[ u(1, \theta) = F(\theta) \quad \text{for} \quad -\pi < \theta < \pi \]

(Hint: \( F(\theta) \) already has the form of a Fourier series on \([−\pi, \pi]\).

The form of the “basic building block” for this problem is \( r^n \cos n\theta \) and \( r^n \sin n\theta \)

Therefore \( u(r, \theta) = 11 + 10r^9 \cos(9\theta) + 8r^7 \sin(7\theta) \) solves the problem.

Give 8pt for this statement and NO PARTIAL CREDIT.
9.  a. 10 pt  Without using Laplace transforms, solve the following IVP:

\[ ty' = 2 + t^3, \quad y'(1) = 3 \]

**ANS.** This is a linear first order ODE. To find the integrating factor we rewrite it as: \[ y' - \frac{2}{y} + t^2 \]

Give 4pt for this statement and NO PARTIAL CREDIT.

The integrating factor is \( t^{-2} \).

Give 2pt for this statement and NO PARTIAL CREDIT.

After multiplying the integrating factor and integrating we have:

\[ t^{-2}y = t + C \]

Give 2pt for this statement.

Plugging in the initial condition gives \( C = 2 \) and the solution is

\[ t^{-2}y = t + 2 \quad \text{or equivalently} \quad y = t^3 + 2t^2 \]

Give 2pt for this statement.

b. 10pt  Assume that \( f(t) \) is a piecewise continuous function with Laplace transform \( \mathcal{L}\{f(t)\} = F(s) \). Derive the following formula for the Laplace transform of \( e^{at}f(t) \):

\[ \mathcal{L}\{e^{at}f(t)\} = F(s - a) \]

**ANS.** The definition of Laplace transform of \( f(t) \) is:

\[ \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t) \, dt \]

Give 3pt for this statement and NO PARTIAL CREDIT.

Therefore replacing \( f(t) \) by \( e^{at}f(t) \) gives:

\[ \mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t) \, dt \]

Give 3pt for this statement and NO PARTIAL CREDIT.

We combine the two exponentials into one:

\[ \mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-(s-a)t}f(t) \, dt \]

Give 2pt for this statement and NO PARTIAL CREDIT.

We recognize the expression on the right as being the definition of the Laplace transform of \( f(t) \) but with the variable \( s \) shifted. i.e. \( F(s - a) \)

Give 2pt for this statement and NO PARTIAL CREDIT.
10. **14pt** Without using Laplace transforms, solve the following IVP:

\[ y'' + 2y' - 3y = t, \quad y(0) = \frac{7}{9}, \quad y'(0) = -\frac{1}{3} \]

**ANS.** The general solution to the above ODE is the sum of the general solution of associated homogeneous eqn \( y'' + 2y' - 3y = 0 \), called \( y_h \) and a particular solution to \( y'' + 2y' - 3y = t \), called \( y_p \) and a particular solution.

Give **4pt** for this statement and NO PARTIAL CREDIT.

Since the characteristic polynomial has roots \( r_1 = 1 \) and \( r_2 = -3 \) we see that \( y_h = c_1 e^t + c_2 e^{-3t} \).

Give **2pt** for this statement and NO PARTIAL CREDIT.

We expect \( y_p \) to be a linear polynomial \( At + B \). Plugging into the ODE gives \(-3B + 2A = 0\) and \(-3A = 1\). ie, \( y_p = -t/3 - 2/9 \).

Give **4pt** for this statement.

We plugin the initial conditions to find \( c_1 \) and \( c_2 \):

\[
\begin{align*}
    c_1 + c_2 - \frac{2}{9} &= \frac{7}{9} \\
    c_1 - 3c_2 - \frac{1}{3} &= -\frac{1}{3}
\end{align*}
\]

Therefore, \( c_1 = \frac{3}{4} \), and \( c_2 = \frac{1}{4} \), \( y = (3e^t + e^{-3t})/4 - t/3 - 2/9 \)

Give **4pt** for correctly finding \( c_1 \) and \( c_2 \). Give **0 pt** for trying to solve a system without the values \( y_p(0) = -\frac{2}{9} \) and \( y_p'(0) = -\frac{1}{3} \) taken into account.