Math 251
November 7, 2005 Ans Key to 2nd Exam

Name ______________________________

There are 8 questions on this exam. Many of them have multiple parts. The point value of each question is indicated either at the beginning of each question or at the beginning of each part where there are multiple part

Where appropriate, **show your work** to receive credit; partial credit may be given.

The use of calculators, books, or notes is not permitted on this exam.

Please turn off your cell phone.

Time limit 1 hour and 15 minutes.

Question		Score
1	26pt	
2	12pt	
3	10pt	
4	10pt	
5	10pt	
6	10pt	
7	12pt	
8	10pt	
Total	100pt	

1. i. 2pt One of the following ODE's represents the displacement $y(t)$ in a spring-mass system with resonance? Circle it.

$y'' + 36y = 4\sin 6t$ $\quad$ $y'' + 36 = 2\sin 6t$ $\quad$ $y'' + 10y' + 34y = 6\cos 3t$ $\quad$ $y'' + 37y' + 36y = \cos 6t$

ANS. Resonance occurs when frequencey of external force equals natural frequency. This happens with the FIRST ODE

ii 4 pt For a spring-mass system system with mass equal to 4 kg, spring contant equal to 9 N/m, which damping constant γ causes critical damping?

ANS. $\gamma_{\text{critical}} = \sqrt{4mk} = \sqrt{4(4)(9)} = 12$

iii. 4 pt For a piecewise continuous function $g(t)$ that has exponential growth, what is the **definition** of the Laplace transform of $g(t)$?

ANS.

$$\mathcal{L}\{\mathrm{g(t)}\} = \int_0^\infty \mathrm{e^{-st} g(t)\, dt}$$

iv 4 pt What is the following Laplace transform: $\quad \mathcal{L}\{te^{2t}\}$

ANS. There are two procedures to the correct answer. Either, since $\mathcal{L}\{\mathrm{t}\} = \frac{1}{\mathrm{s}^2}$, $\mathcal{L}\{\mathrm{e^{2t}t}\} = \frac{1}{(\mathrm{s}-2)^2}$.
Or, since $\mathcal{L}\{\mathrm{e^{2t}}\} = \frac{1}{\mathrm{s}-2}$, $\mathcal{L}\{\mathrm{te^{2t}}\} = -\frac{-1}{(\mathrm{s}-2)^2}$.

Give **2pt** for writing correct starting point. **2pt** for correct answer. **Do not give** more than **2pt** for correct answer without correct starting point!

v. 4 pt Suppose that the Laplace transform of y is Y. If $y(0) = 2$ and $y'(0) = -3$, then find the Laplace transform of y''.

ANS. Give **1pt** for $\mathcal{L}\{\mathrm{y'}\} = \mathrm{sY} - 2$
Give **3pt** for $\mathcal{L}\{\mathrm{y''}\} = \mathrm{s(sY-2)} - (-3) = \mathrm{s^2Y - 2s + 3}$

vi. 4 pt Suppose that a force of $-3\delta(t-1)$ acts on an object of mass 1 kg. If $y(0) = 2$ and $y'(0) = 2$, then find $y'(2)$.

ANS. This force decrease momentum (equal to velocity, here) by 3 units at $t = 1$. It does nothing at any other time. So $y'(2) = 2 - 3 = -1$ (There is no way to earm partial credit here).

vii 4 pt Suppose that the homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ has only one eigenvalue $r_1 = 3$ and that all nonzero vectors eigenvectors. Find the solution satifying the IVP: $\mathbf{x}(0) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

ANS. Give **2pt** for indicating the solution is a multiple of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Give another **2pt** for if the multiple is correct: ie, $\mathbf{x} = e^{3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

2. In parts **a - c** determine the form of a particular solution y_p having the **least** number of unknown constants. **DO NOT DETERMINE** the unknown constants appearing in your answers in parts **a** and **b**.

a. 2pt $y'' - 3y' + 2y = (2t^2 - 1)e^{6t}$

ANS. The roots of the characteristic polynomial are 1 and 2.
Therefore the solution is a generic polynomial of degree 2 times the same exponential.
Give **2pt** and no partial credit possible here.

b. 2pt $y'' - 2y' + y = (3t^2 + 2t)e^t$

ANS. The roots of the characteristic polynomial are 1 and 1. Therefore the solution is a generic polynomial of degree 2 times the same exponential times t^2.
Give **2pt**.
Give **1pt** for the following answer a generic polynomial of degree 4 times the same exponential. Otherwise, no partial credit possible here.

c. 2pt $y'' - 3y' + 2y = (2t^2 + 3)e^t$

ANS. The roots of the characteristic polynomial are 1 and 2. Therefore the solution is a generic polynomial of degree 2 times the same exponential times t^1.
Give **2pt**.
Give **1pt** for the following answer a generic polynomial of degree 4 times the same exponential. Otherwise, no partial credit possible here.

d. 6pt Without using Laplace transforms, find a particular solution to the following ODE:

$$y'' - 2y' + y = \sin 3t$$

(In this part you need to **determine the unknown constant(s) in the solution.**

ANS. There are two possible solutions.

The roots of the characteristic polynomial are 1 and 1.

First solution: Complexify the ODE/

$$y'' - 2y' + y = e^{3it}$$

and find $y_C = Ae^{3it}$ which solves the ODE: Give 2pts

So we plug y_C into the ODE.

$$\begin{aligned} (y_C &= Ae^{3it}) \\ -2(y'_C &= A3ie^{3it}) \\ (y''_C &= A(-9)e^{7t}) \end{aligned}$$

So A_C is determined by $A(-8-6i) = 1$, or $A = \frac{1}{2(-4-3i)} = \frac{-4+3i}{50}$ Give 3pts for A correctly determined, give 1pts or 2pts if there are errors but correct idea is discernable.

Finally take imaginary part of $y_C = \frac{1}{50}(-4+3i)(\cos 3t + i \sin 3t)$. This is $y_p = \frac{1}{50}(3\cos 3t - 4\sin 3t)$ Give 2pts for this.

Second solution: Plug $y_p = A\cos 3t + B\sin 3t$ into the ODE. Give 2pts

$$\begin{aligned} (y_p &= A\cos 3t + B\sin t \\ -2(y'_p &= -3A\sin 3t + 3B\cos 3t \\ (y''_p &= -9A\cos 3t - 9B\sin 3t \end{aligned}$$

Equating corresponding coefficients of sine and cosine we obtain we obtain
$-8B + 6A = 1$ and $-8A - 6B = 0$.
Give 3pts for these two equations correctly determined; give 1pts or 2pts if there are errors but correct idea is discernable.

Solving these two equation gives $A = 3/50$ and $B = -4/50$.
Give 1pt for this.

3. An object with mass 3 kg stretches a spring 10/9 meters to its equilibrium position. Assume that g, the acceleration due to gravity, is 10 meter/sec^2 and that there is no damping device attached. Also assume that at time $t = 0$ the object is released 3 meter below its equilibrium position with a upward velocity of 12 meter/sec.

a. 2pt Write down a differential equation for $y(t)$ with initial conditions for the displacement of the object from its equilibrium position?

ANS. $3y'' + 27y = 0$ and $y(0) = 3$, $y'(0) = -12$

Here, give **1/2 pt** for each of $m = 3$ $k = 27$ $y(0) = 3$, $y'(0) = -12$, and truncating the total to an integer.

b. 4pt Find the solution $y(t)$ of the above IVP.

ANS. The general solutions is $y = c_1 \cos 3t + c_2 \sin 3t$. Give 2pts for this.

Solving the IVP: $c_1 = 3$, $c_2 = -4$. Give 2pts for this.

c. 2pt Find the amplitude of $y(t)$.

ANS. $R = \sqrt{c_1^2 + c_2^2} = 5$. Give 2pts for this.

d. 1pt Find the natural frequency of the above spring mass system.

ANS. $\omega_0 = 3$ Give 1pt for this.

e. 1pt If an external force equal to $\sin \omega t$ is added to the above spring mass system, then what value of ω would cause resonance?

ANS. $\omega = 3$ Give 1pt for this.

4. **a.** **4pt** Find the function $f(t)$ whose Laplace transform is equal to

$$\frac{s}{s^2+4s+3}$$

ANS. Factor the denominator and use partial fractions to write : $\frac{3}{2}s+3-\frac{1}{2}(s+1)$ Give **2pt**. Give **1pt** if only arithmetic error.

So

$$\frac{s}{s^2+4s+3} = \mathcal{L}\{\frac{3}{2}\mathrm{e}^{-3\mathrm{t}} - \frac{1}{2}\mathrm{e}^{-\mathrm{t}}\}$$

Give **2pt**. Obviously an non arithmetic error in the first part can throw the answer out of the ball park. If there is something related to inverse Laplace transforms that is recognizably correct then give **1pt** for the entire part.

Also, there may be some who give the answer

$$e^{-2t}(\cosh t - 2\sinh t)$$

which is correct.

b. **6 pt** Find the function $f(t)$ whose Laplace transform is equal to

$$\frac{s}{s^2+4s+40}$$

ANS. First complete the square:

$$\frac{s}{(s+2)^2+6^2}$$

Give **2pt**. Give **1pt** if only arithmetic error.

Need to write this as

$$\frac{s+2}{(s+2)^2+6^2} - \frac{2}{6}\frac{6}{(s+2)^2+6^2}$$

Give **2pt**.

So by the shift formula and sine and cosine formulas:

$$\frac{s}{(s+2)^2+6^2} = \mathcal{L}\{\mathrm{e}^{4\mathrm{t}}\left(\cos 6\mathrm{t} - \frac{1}{3}\sin 6\mathrm{t}\right)\}$$

Give **2pt**. Obviously an non arithmetic error in the first part or not recognizing the need for the second step can throw the answer out of the ball park. If there is something related to inverse Laplace transforms that is recognizably correct then give **2pt** for the entire part.

5. a. 5 pt Consider the function

$$f(t) = \begin{cases} 0, & \text{if} \quad t < 1 \\ t-1, & \text{if} \quad 1 \le t < 2 \\ 1, & \text{if} \quad 2 \le t \end{cases}$$

Sketch a graph of this function and find a formula for $f(t)$ in terms of unit step functions $u(t-c)$, for appropriate values of c.

Give **1pt** for a correct graph.

Give **3pt** more points $(t-1)(u(t-2)-u(t-1))$ exists somewhere in the answer. If it does not but $u(t-2)-u(t-1)$ does, then give only **1pt** more point.

Finally, give **1pt** more (ie, a total of **4pts** for answering the nongraphing part if $(t-1)(u(t-2)-u(t-1))+u(t-1)$ is given as the final answer.

b. 5 pt Find the Laplace transform of the function $e^t u(t-2)$

ANS. There are two procedures to the correct answer.

First: Since $\mathcal{L}\{\mathrm{u(t-2)}\} = \frac{\mathrm{e}^{-2\mathrm{s}}}{\mathrm{s}}$, $\mathcal{L}\{\mathrm{e^t u(t-2)}\} = \frac{\mathrm{e}^{-2(\mathrm{s}-1)}}{\mathrm{s}-1}$.
Give **2pt** for writing the first formula correctly somewhere in the procedure.
Give **5pt** for correct answer.
Do not give more than **2pt** if this is the procedure that is being attempted if the first formula is not written.

The second procedure: since $\mathcal{L}\{\mathrm{e^t u(t)}\} = \frac{1}{\mathrm{s}-1}$, $\mathcal{L}\{\mathrm{e^{t-2} u(t-2)}\} = \frac{\mathrm{e}^{-2\mathrm{s}}}{\mathrm{s}-1}$.
Give **2pt** for writing the first formula correctly somewhere in the procedure.

Multiplying this through by e^2 gives the correct answer:
$\mathcal{L}\{\mathrm{e^t u(t-2)}\} = \frac{\mathrm{e}^2\mathrm{e}^{-2\mathrm{s}}}{\mathrm{s}-1}$. (This of course is the same as the previous answer.)
Give **5pt** for correct answer.
Give **4pt** if answer is correct execpt for multiplying by e^2
Do not give more than **2pt** if this is the procedure that is being attempted and if the first formula is not written.

6. 10 pt Use Laplace Transforms to solve the following

$$y' + 3y = tu(t-2) + \delta(t-2) \qquad y(0) = 1,$$

ANS. Take Laplace of each term on the right

$$\mathcal{L}\{\mathrm{tu(t-2)}\} = -\frac{\mathrm{d}}{\mathrm{ds}}\frac{\mathrm{e}^{-2s}}{\mathrm{s}} = \mathrm{e}^{-2s}\left(\frac{1}{\mathrm{s}^2} - 2\frac{1}{\mathrm{s}}\right)$$

Give **3pts** for this.

$$\mathcal{L}\{\delta(\mathrm{t}-2)\} = \mathrm{e}^{-2s}$$

Give **1pts** for this.

$$\mathcal{L}\{\mathrm{y}' + 3\mathrm{y}\} = \mathrm{sY} - 1 + 3\mathrm{Y}$$

Give **1pts** for this.

Summarizing what has been done so far together:

$$Y = \frac{1}{s+3} + e^{-2s}\left(\frac{1}{s+3} + 2\frac{1}{s(s+3)} - \frac{1}{s^2(s+3)}\right)$$

The inverse Laplace of the first term on the right is given by

$$\mathcal{L}\{\mathrm{e}^{-3t}\} = \frac{1}{\mathrm{s}+3}$$

Give an easy **1pt** for this.

By partial fractions

$$\frac{1}{s+3} + 2\frac{1}{s(s+3)} - \frac{1}{s^2(s+3)} = \frac{2/9}{s+3} + \frac{7/9}{s} - \frac{1/3}{s^2}$$

Give **2pts** for this.

Therefore,

$$\mathcal{L}\left\{u(t)\left(\frac{2}{9}e^{-3t} + \frac{7}{9} - \frac{1}{3}t\right)\right\} = \frac{2/9}{s+3} + \frac{7/9}{s} - \frac{1/3}{s^2}$$

Give **1pts** for this.

Finally,

$$\mathcal{L}\left\{u(t-2)\left(\frac{2}{9}e^{-3(t-2)} + \frac{7}{9} - \frac{1}{3}(t-2)\right)\right\} = e^{-2s}\left(\frac{2/9}{s+3} + \frac{7/9}{s} - \frac{1/3}{s^2}\right)$$

Give **1pts** for this.

7. In each part of this Problem do the following:
i. Sketch a phase portrait for this system.
ii. State the name associated with the critical point at $(0, 0)$ and state whether it is stable, asymptotically stable or unstable?

a. 4pt The homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ whose general solution is:
$\mathbf{x} = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

ANS. ii. The eigenvalues are both negative.
1 pt for stating the critical pt origin is a node. **1 pt** for stating the critical pt origin is asymptotically stable. Do **NOT** give any credit if the word "asymptotically" is omitted.

i. 1 pt for drawing 4 trajectories which are at the pts corresponding to the two eigenvectors and their negatives with correct orientations.

1 pt for drawing 4 more trajectories correctly. All trajectories must be drawn tangent to the direction of $\pm \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for large t.

b. 4pt The homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ whose general solution is:
$\mathbf{x} = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

ANS. ii. The eigenvalues have opposite signs.
1 pt for stating the critical pt origin is a saddle. **1 pt** for stating the critical pt origin is unstable.

i. 1 pt for drawing 4 trajectories which are at the pts corresponding to the two eigenvectors and their negatives with correct orientations.
1 pt for drawing 4 more trajectories correctly.

c. 4pt The homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ whose general solution is:
$\mathbf{x} = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

ANS. ii. A has only one positive eignevalue.
1 pt for stating the critical pt origin is a **proper** node. No credit if the word **proper** is omitted.
1 pt for stating the critcal pt origin is unstable.

i. 1 pt for drawing 4 trajectories which are at the pts corresponding to the two eigenvectors and their negatives with correct orientations.
1 pt for drawing 1 more trajectories correctly.
If the 5th ray directed away from the origin is **not** present a statement saying that all trajectories are rays directed away from the origin is **not** present, then do **not** give full credit for this part.

8. **a. 6pt** Find the general solution of $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$.

ANS. The characteristic polynomial is $r^2 + 9$ and hence the eigenvalues are $\pm 3i$ Give **1 pt**

A complex eigenvector is $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ Give **1 pt**

A complex solution to this system is $\mathbf{x}_C = (\cos 3t + i \sin 3t) \begin{pmatrix} 1 \\ -i \end{pmatrix}$ Give **1 pt**

Two real solutions which are not multiples of each other are obtained by taking real and imaginary parts of $\mathbf{x}_C$

$\mathbf{x}_1 = \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}$ Give **1 pt**

$\mathbf{x}_2 = \begin{pmatrix} \sin 3t \\ -\cos 3t \end{pmatrix}$ Give **1 pt**

$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$Give **1 pt**

b. 2pt Sketch a phase portrait for the system given in Part a.

ANS. One trajectory consisting of a circle (or ellipse) is sufficient here. Give **1 pt**

Its orientation is counter clockwise. Give **1 pt**

Do not give full credit for this part unless some statement indicating how the orientation was determined is given.

c. 1pt What is the critical point called?

ANS. Center. Give **1 pt**

Do not accept circle or ellipse for the answer to this part.

d. 1pt What is its stability?

ANS. Stable. Give **1 pt**

Do not give credit if the word “asymptotically” appears here!