Math 251
October 12, 2005 ANSWER KEY to First Exam

NAME:______________________________ Section #:__________

There are 9 questions on this exam. Many of them have multiple parts. The point value of each question is indicated either at the beginning of each question or at the beginning of each part where there are multiple part

Show all your work. Partial credit may be given.

The use of calculators, books, or notes is not permitted on this exam.

Please turn off your cell phone before starting this exam.

Time limit 1 hour and 15 minutes.

Question		Score
1	24pt	
2	10pt	
3	10pt	
4	8pt	
5	12pt	
6	10pt	
7	10pt	
8	8pt	
9	10pt	
Total	102pt	

1. **a. 2pt** Consider the following differential equation $y' = y + 2t$. Without solving it, determine the slope of the tangent line to the solution at the point $(1, 2)$.

ANS. 2pt for plugging $(1, 2)$ into $f(t, y)$ gives $2 + 2(1) = 4$.
Give **1pt** for $1 + 2(2) = 5$.

b. 2pt Find the Wronskian $W(y_1, y_2)$ of the functions $y_1 = \sin t$ and $y_2 = \cos t$

ANS. 2pt for $\sin^2 t + \cos^2 t$.
1pt for $\sin^2 t - \cos^2 t$.

c. 2pt Suppose y_1 and y_2 are two solutions of the ODE $y'' + (\sin t)y' + y = 0$. and suppose that their Wronskian by $W(y_1, y_2)(t)$ is 2 at $t = 0$. Find $W(y_1, y_2)(t)$ for any t.

ANS. 2pt for $(2/e)e^{\cos t}$. **1pt** for anything else having the form $?e^{\cos t}$.

For the initial value problems in parts **d.** through **g.** state whether or not one of our two existence and uniqueness theorems for first order ODE's guarantees a unique solution. If the answer is yes and the theorem provides an interval of existence, then state what the interval is **without actually solving the equation**.

d. 2pt __________________ $(t + 2)y' + (y - 1)^{2/3} = 0, \quad y(3) = 0$

ANS. 2pt unique solution guaranteed
Remove 1pt if interval is given.

e. 2pt __________________ $(t + 2)y' + (y - 1)^{2/3} = 0, \quad y(0) = 1$

ANS. 2pt no guarantee applies

f. 2pt __________________ $(t^2 + 2t)y' + y = 0, \quad y(-3) = 0$

ANS. 2pt unique solution guaranteed to exist on $(-\infty, -2)$

g. 2pt __________________ $(t^2 + 2t)y' + y = 0, \quad y(3) = 0$

ANS. 2pt unique solution guaranteed to exist on $(0, \infty)$

In parts **h. 2pt** through **j.** assume that $L[y] = y'' + p(t)y' + q(t)y$ and that $p(t)$ and $q(t)$ are continuous functions on the entire real axis $(-\infty, \infty)$.

h. 2pt Circle a pair of functions among the following functions which could be a fundamental set of solutions for the differential equation $L[y] = 0$. (There is more than one correct answer.)

$$y_1 = e^{3+t/2}, \qquad y_2 = e^{3+2t}, \qquad y_3 = e^{2t-2}, \qquad y_4 = 0$$

ANS. 2pt Either y_1 and y_2 or y_1 and y_3

j. 2pt Suppose that y_1 and y_2 have the following properties: $L[y_1] = t$, and $L[y_2] = 0$. Then one of the following solves the differential equation $L[y] = 2t$. Circle it.

$$y_1/2 + y_2 \qquad 2y_1 + 2y_2 \qquad 2y_2 \qquad y_1 + y_2$$

ANS. 2pt $L[2y_1 + 2y_2] = 2t$

In parts **k.** through **m.** match the ODE's on the left with a description on the right.

ANS. 2pt ii k. 2pt $y' = e^{y+t}$

ANS. 2pt i l. 2pt $y' = 2 - y$

ANS. 2pt ii m. 2pt $y' = t^2 - 3y$

i. linear and separable

ii. linear but not separable

iii. separable but not linear

iv. not separable and not linear

2. Consider the autonomous differential equation

$$y' = (y-6)(2-y) = -y^2 + 8y - 12$$

a. 2pts Sketch a direction field for this equation. Indicate the equilibrium solutions in your sketch.

ANS. Use the Direction Field applet to sketch the direction field.

b. 2pts Which equilibrium solution is (are) **asymptotically stable** and which is (are) **unstable**.

ANS. 1pt $y = 6$ is asymptotically stable
1pt $y = 2$ is unstable.

c. 3pts Find a formula for y'' in terms of y.

ANS. 3pt $y'' = (-2y+8)(y-6)(2-y)$
Give no credit for answer $y'' = -2y + 8$

e. 3pts Sketch the graph of the solution with $t \geq 0$ with the initial value $y(0) = 3$ indicating its concavity as accurately as possible.

ANS. 3pt Check for inflection point at intersection of graph with $y = 4$

3. 10 pt Find the general solution to the differential equation

$$ty' = t^6 + 5y \qquad\qquad t > 0$$

ANS. u**4pt** Recognize that ODE is linear and correctly identify integrating factor: $\mu = t^{-5}$

3pt Multiply through by μ: $(yt^{-5})' = 1$

3pt Integrate both sides wrt t: $yt^{-5} : t + C$ or $y = t^6 + Ct^5$

Remove 2pt for omitting the constant of integration.

4 a. 6pts Find the general solution to the following IVP

$$3y' = 4y^4t^3$$

ANS. 2pt Recognize that ODE is separate and separate correctly: $3\frac{y'}{y} = 4t^3$

2pt Integrate both sides wrt t: $3\ln|y| = t^4 + C$

2pt Exponential $|y|^3 = C_1e^{t^4}$, C_1 is a positive constant.

2pt Note that solution $y = 0$ was **lost** in the separation process. Thus $y^3 = C_2e^{t^4}$, C_2 is any constant, negative, positive, zero.

b. 2pts Find the solution to the above differential equation that satisfies the initial condition $y(1) = 2$

ANS. 2pt Set $t = 1$, $y = 2$ to determine C_2: $y^3 = (8/e)e^{t^4}$, C_2 is any constant,

5. **a. 6pt** Find the general solution the following differential equation with initial conditions:

$$y'' + 6y' + 13y = 0$$

ANS. 3pt The characteristic polynomial $r^2 + 6r + 13$ has complex roots $-3 \pm 2i$.

3pt The general solution is $y = e^{-3t}(c_1 \cos 2t + c_1 \sin 2t)$. Accept the complex general solution temporarily!

b. 4pt Solve the following IVP for the above ODE:

$$y(0) = 1, \qquad y'(0) = 3$$

ANS. 2pt We need a formula for y': $y' = -3y + 2e^{-3t}(-c_1 \sin 2t + c_2 \cos 2t)$.

ANS. 2pt Set $t = 0$ in y and y': $1 = c_2$ and $3 = -3 + 2c_2$. So $y = e^{-3t}(\cos 2t + 3 \sin 2t)$.

A complex general solution used correctly will also yield this answer! If a complex general solution does not yield the correct answer here, then remove 2pts here and 2pts from the previous part of this problem.

c. 2pt Solve the following IVP for the above ODE:

$$y(999) = 1, \qquad y'(999) = 3$$

ANS. 2pt $y(t - 999)$.

6. a. 2pt One of the following differential equations is exact. Circle it:

$$2t + 3y + (2y + 3t)y' = 0 \qquad\qquad 2t + 3y + (2t + 3y)y' = 0$$

ANS. The first ODE is exact.

2pt

b. 8pt Find the solution to the following differential equation which satisfies $y(1) = 0$:

$$y^2 + t^2 + (e^{2y} + 2ty)y' = 0$$

ANS. 2pt We seek an F such that $F_t = y^2 + t^2$. Hence $F(t, y) = ty^2 + t^3/3 + h(y)$

2pt To determine $h(y)$ we use $F_y = 2ty + \frac{dh}{dy}$

2pt We see $\frac{dh}{dy} = e^{2y}$, $h(y) = (1/2)e^{2y}$.

2pt We conclude $F(t, y) = ty^2 + t^3/3 + (1/2)e^{2y} = C$ is the general solution. Setting $t = 1$ and $y = 0$ gives $C = 5/6$, $ty^2 + t^3/3 + (1/2)e^{2y} = 5/6$

7. A tank with a capacity of 200 liters initially contains a mixture of 25 grams of salt disolved in 50 liters of water. A salt water mixture with a concentration of 2 grams/liter enters the tank at the rate of 6 liters/min. Well stirred mixture leaves the tank at 6 liters/min.

a. 4pt Let $Q(t)$ be the quantity of salt in the tank at time $t \geq 0$. Write down a differential equation and an initial condition for the quantity $Q(t)$ of salt in grams in the tank at any time $t \geq 0$. **(Do not solve it.)**

ANS. 3pt $Q' = 2(6) - \frac{Q}{50}6$
1pt $Q(0) = 25$

b. 2pt Draw a direction field for the ODE.

Use the Direction Field applet to sketch the direction field.

c. 2pt Without solving the ODE, determine approximately the quantity of salt in the tank after a long time.

ANS. 2pt 100grams

d. 2pt Without solving the ODE, determine $Q'(0)$.

ANS. 2pt $Q'(0) = 2(6) - \frac{Q(0)}{50}6 = 12 - \frac{1}{2}6 = 9$grams/min

8. Match the differential equations listed below with the descriptions of long time behavior listed below. (Each description matches only one equation. Please place your answer in the space provided.)

2pt II. **a.** **2pt** $y'' - y' - 2y = 0$

2pt I. **b.** **2pt** $y'' + 4y' + 4y = 0$

2pt V. **c.** **2pt** $y'' - 4y' + 29y = 0$

2pt VI. **d.** **2pt** $y'' + y = 0$

I. Every solution approaches 0 as $t \to \infty$

II. Has a nonzero solution that approaches 0 as $t \to \infty$ and has a nonzero solution that approaches ∞ as $t \to \infty$

III. Every nonzero solution approaches either ∞ or $-\infty$ as $t \to \infty$

IV. Every nonzero solution has oscillations which become progressively larger as $t \to \infty$

V. Every nonzero solution has oscillations which become progressively smaller as $t \to \infty$

VI. Every nonzero solution oscillates with constant amplitude $t \to \infty$.

9. 10pt The ODE

$$t^2y'' - ty' + y = 0, \qquad t > 0$$

obviously has a solution $y_1 = t$. Use the method of reduction of order to find another solution of this linear homogeneous ODE that is not a constant multiple of y_1.

ANS.

2pt We seek a solution y_2 of the form $y_2 = vt$, where v is a constant function.

So we plug in:

2pt $y_2' = v't + v$ and $y_2'' = v''t + 2v'$.

and obtain:

2pt $t^2y'' - ty' + y = t^2(v''t + 2v') - t(v't + v) + vt = 0$ Or, more simply, $v''t^3 + v't^2 = 0$

2pt dividing by t^2 gives $v''t + v' = 0$.

2pt This is a separable first order ODE for v'. We see that $v''/v' = -1/t$. Thus, $\ln v' = -lnt$ $v' = 1/t$ and $v = \ln t$ and $y_2 = t \ln t$