Second Order Linear Partial Differential Equations

Part IV

One-dimensional undamped wave equation; D’Alembert solution of the wave equation; damped wave equation and the general wave equation; two-dimensional Laplace equation

The second type of second order linear partial differential equations in 2 independent variables is the one-dimensional wave equation. Together with the heat conduction equation, they are sometimes referred to as the “evolution equations” because their solutions “evolve”, or change, with passing time. The simplest instance of the one-dimensional wave equation problem can be illustrated by the equation that describes the standing wave exhibited by the motion of a piece of undamped vibrating elastic string.
Undamped One-Dimensional Wave Equation: Vibrations of an Elastic String

Consider a piece of thin flexible string of length $L$, of negligible weight. Suppose the two ends of the string are firmly secured (“clamped”) at some supports so they will not move. Assume the set-up has no damping. Then, the vertical displacement of the string, $0 < x < L$, and at any time $t > 0$, is given by the displacement function $u(x, t)$. It satisfies the homogeneous one-dimensional undamped wave equation:

$$a^2 u_{xx} = u_{tt}$$

Where the constant coefficient $a^2$ is given by the formula $a^2 = T/\rho$, such that $a =$ horizontal propagation speed (also known as phase velocity) of the wave motion, $T =$ force of tension exerted on the string, $\rho =$ mass density (mass per unit length). It is subjected to the homogeneous boundary conditions

$$u(0, t) = 0, \text{ and } u(L, t) = 0, \quad t > 0.$$

The two boundary conditions reflect that the two ends of the string are clamped in fixed positions. Therefore, they are held motionless at all time.

The equation comes with 2 initial conditions, due to the fact that it contains the second partial derivative of time, $u_{tt}$. The two initial conditions are the initial (vertical) displacement $u(x, 0)$, and the initial (vertical) velocity $u_t(x, 0)^*$, both are arbitrary functions of $x$ alone. (Note that the string is merely the medium for the wave, it does not itself move horizontally, it only vibrates, vertically, in place. The resulting undulation, or the wave-like “shape” of the string, is what moves horizontally.)

* Velocity = rate of change of displacement with respect to time. The other first partial derivative $u_x$ represents the slope of the string at a point $x$ and time $t$. 

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One-dimensional Homogeneous undamped wave equation

\[ a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0, \]

We first let \( u(x, t) = X(x)T(t) \) and separate the wave equation into two ordinary differential equations. Substituting \( u_{xx} = X'' T \) and \( u_{tt} = XT'' \) into the wave equation, it becomes

\[ a^2 X'' T = XT''. \]
Dividing both sides by $a^2 X T$:

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

As for the heat conduction equation, it is customary to consider the constant $a^2$ as a function of $t$ and group it with the rest of $t$-terms. Insert the constant of separation and break apart the equation:

$$\frac{X''}{X} = -\lambda \quad \rightarrow \quad X'' = -\lambda X \quad \rightarrow \quad X'' + \lambda X = 0,$$

$$\frac{T''}{a^2 T} = -\lambda \quad \rightarrow \quad T'' = -a^2 \lambda T \quad \rightarrow \quad T'' + a^2 \lambda T = 0.$$  

The boundary conditions also separate:

$$u(0, t) = 0 \quad \rightarrow \quad X(0)T(t) = 0 \quad \rightarrow \quad X(0) = 0 \quad \text{or} \quad T(t) = 0$$

$$u(L, t) = 0 \quad \rightarrow \quad X(L)T(t) = 0 \quad \rightarrow \quad X(L) = 0 \quad \text{or} \quad T(t) = 0$$

As usual, in order to obtain nontrivial solutions, we need to choose $X(0) = 0$ and $X(L) = 0$ as the new boundary conditions. The result, after separation of variables, is the following simultaneous system of ordinary differential equations, with a set of boundary conditions:

$$X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(L) = 0,$$

$$T'' + a^2 \lambda T = 0.$$
The next step is to solve the eigenvalue problem

\[ X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0. \]

We have already solved this eigenvalue problem, recall. The solutions are

**Eigenvalues:** \[ \lambda = \frac{n^2 \pi^2}{L^2}, \quad n = 1, 2, 3, \ldots \]

**Eigenfunctions:** \[ X_n = \sin \frac{n \pi x}{L}, \quad n = 1, 2, 3, \ldots \]

Next, substitute the eigenvalues found above into the second equation to find \( T(t) \). After putting eigenvalues \( \lambda \) into it, the equation of \( T \) becomes

\[ T'' + a^2 \frac{n^2 \pi^2}{L^2} T = 0. \]

It is a second order homogeneous linear equation with constant coefficients. It’s characteristic have a pair of purely imaginary complex conjugate roots:

\[ r = \pm \frac{an \pi}{L} i. \]

Thus, the solutions are simple harmonic:

\[ T_n(t) = A_n \cos \frac{an \pi t}{L} + B_n \sin \frac{an \pi t}{L}, \quad n = 1, 2, 3, \ldots \]

Multiplying each pair of \( X_n \) and \( T_n \) together and sum them up, we find the general solution of the one-dimensional wave equation, with both ends fixed, to be
There are two sets of (infinitely many) arbitrary coefficients. We can solve for them using the two initial conditions.

Set $t = 0$ and apply the first initial condition, the initial (vertical) displacement of the string $u(x, 0) = f(x)$, we have

$$u(x,0) = \sum_{n=1}^{\infty} \left( A_n \cos(0) + B_n \sin(0) \right) \sin \frac{n \pi x}{L}$$

$$= \sum_{n=1}^{\infty} A_n \sin \frac{n \pi x}{L} = f(x)$$

Therefore, we see that the initial displacement $f(x)$ needs to be a Fourier sine series. Since $f(x)$ can be an arbitrary function, this usually means that we need to expand it into its odd periodic extension (of period $2L$). The coefficients $A_n$ are then found by the relation $A_n = b_n$, where $b_n$ are the corresponding Fourier sine coefficients of $f(x)$. That is

$$A_n = b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n \pi x}{L} \, dx.$$

Notice that the entire sequence of the coefficients $A_n$ are determined exactly by the initial displacement. They are completely independent of the other sequence of coefficients $B_n$, which are determined solely by the second initial condition, the initial (vertical) velocity of the string. To find $B_n$, we differentiate $u(x, t)$ with respect to $t$ and apply the initial velocity, $u_t(x, 0) = g(x)$.
\[ u_t(x, t) = \sum_{n=1}^{\infty} \left( -A_n \frac{an\pi}{L} \sin \frac{an\pi t}{L} + B_n \frac{an\pi}{L} \cos \frac{an\pi t}{L} \right) \sin \frac{n\pi x}{L} \]

Set \( t = 0 \) and equate it with \( g(x) \):

\[ u_t(x, 0) = \sum_{n=1}^{\infty} B_n \frac{an\pi}{L} \sin \frac{n\pi x}{L} = g(x) \cdot \]

We see that \( g(x) \) needs also be a Fourier sine series. Expand it into its odd periodic extension (period \( 2L \)), if necessary. Once \( g(x) \) is written into a sine series, the previous equation becomes

\[ u_t(x, 0) = \sum_{n=1}^{\infty} B_n \frac{an\pi}{L} \sin \frac{n\pi x}{L} = g(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \]

Compare the coefficients of the like sine terms, we see

\[ B_n \frac{an\pi}{L} = b_n = \frac{2}{L} \int_{0}^{L} g(x)\sin \frac{n\pi x}{L} \, dx \cdot \]

Therefore,

\[ B_n = \frac{L}{an\pi} b_n = \frac{2}{an\pi} \int_{0}^{L} g(x)\sin \frac{n\pi x}{L} \, dx \cdot \]

As we have seen, half of the particular solution is determined by the initial displacement, the other half by the initial velocity. The two halves are determined independent of each other. Hence, if the initial displacement \( f(x) = 0 \), then all \( A_n = 0 \) and \( u(x, t) \) contains no sine-terms of \( t \). If the initial velocity \( g(x) = 0 \), then all \( B_n = 0 \) and \( u(x, t) \) contains no cosine-terms of \( t \).
Let us take a closer look and summarize the result for these 2 easy special cases, when either \( f(x) \) or \( g(x) \) is zero.

**Special case I:** Nonzero initial displacement, zero initial velocity: \( f(x) \neq 0, \quad g(x) = 0 \).

Since \( g(x) = 0 \), then \( B_n = 0 \) for all \( n \).

\[
A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 1, 2, 3, \ldots
\]

Therefore,

\[
u(x, t) = \sum_{n=1}^{\infty} A_n \cos \frac{an\pi t}{L} \sin \frac{n\pi x}{L}.
\]

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The D’Alembert Solution

In 1746, Jean D’Alembert† produced an alternate form of solution to the wave equation. His solution takes on an especially simple form in the above case of zero initial velocity.

Use the product formula \( \sin(A) \cos(B) = \left[ \sin(A - B) + \sin(A + B) \right] / 2 \), the solution above can be rewritten as

\[
 u(x, t) = \frac{1}{2} \sum_{n=1}^{\infty} A_n \left( \sin \left( \frac{n\pi (x - at)}{L} \right) + \sin \left( \frac{n\pi (x + at)}{L} \right) \right)
\]

Therefore, the solution of the undamped one-dimensional wave equation with zero initial velocity can be alternatively expressed as

\[
 u(x, t) = [F(x - at) + F(x + at)] / 2.
\]

In which \( F(x) \) is the odd periodic extension (period \( 2L \)) of the initial displacement \( f(x) \).

An interesting aspect of the D’Alembert solution is that it readily shows that the starting waveform given by the initial displacement would keep its general shape, but it would also split exactly into two halves. The two halves of the wave form travel in the opposite directions at the same finite speed of propagation \( a \). This can be seen by the fact that the two halves of the wave form, in terms of \( x \), are being translated/moved in the opposite direction, to the right and left, in the form of phase shifts, at the rate of distance \( a \) units per unit time. Hence the value \( a \) is also known as the wave’s phase velocity.

† Jean le Rond d’Alembert (1717 – 1783) was a French mathematician and physicist. He is perhaps best known to calculus students as the inventor of the Ratio Test for convergence.
Furthermore, once the “wave front” has passed over a point on the string, the displacement at that point will be restored to its previous state before the arrival of the wave. In physics, this aspect of a clearly-defined, echo-less, wave motion of a one-dimensional wave is called the Huygens’ Principle. (The principle also holds for solutions of a three-dimensional wave equation. But it is not true for two-dimensional waves.)

Special case II: Zero initial displacement, nonzero initial velocity: \( f(x) = 0, \ g(x) \neq 0. \)

Since \( f(x) = 0, \) then \( A_n = 0 \) for all \( n. \)

\[
B_n = \frac{2}{an \pi} \int_0^L g(x) \sin \frac{n \pi x}{L} \, dx, \quad n = 1, 2, 3, \ldots
\]

Therefore,

\[
u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{an \pi t}{L} \sin \frac{n \pi x}{L}.
\]
**Example:** Solve the one-dimensional wave problem

\[ 9 u_{xx} = u_{tt}, \quad 0 < x < 5, \quad t > 0, \]
\[ u(0, t) = 0, \text{ and } u(5, t) = 0, \]
\[ u(x, 0) = 4\sin(\pi x) - \sin(2\pi x) - 3\sin(5\pi x), \]
\[ u_t(x, 0) = 0. \]

First note that \( a^2 = 9 \) (so \( a = 3 \)), and \( L = 5 \).

The general solution is, therefore,

\[ u(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos \frac{3n\pi t}{5} + B_n \sin \frac{3n\pi t}{5} \right) \sin \frac{n\pi x}{5}. \]

Since \( g(x) = 0 \), it must be that all \( B_n = 0 \). We just need to find \( A_n \). We also see that \( u(x, 0) = f(x) \) is already in the form of a Fourier sine series. Therefore, we just need to extract the corresponding Fourier sine coefficients:

\[ A_5 = b_5 = 4, \]
\[ A_{10} = b_{10} = -1, \]
\[ A_{25} = b_{25} = -3, \]
\[ A_n = b_n = 0, \quad \text{for all other } n, n \neq 5, 10, \text{ or } 25. \]

Hence, the particular solution is

\[ u(x, t) = 4\cos(3\pi t) \sin(\pi x) - \cos(6\pi t) \sin(2\pi x) - 3\cos(15\pi t) \sin(5\pi x). \]
We can also solve the previous example using D’Alembert’s solution. The problem has zero initial velocity and its initial displacement has already been expanded into the required Fourier sine series, \( u(x, 0) = 4\sin(\pi x) - \sin(2\pi x) - 3\sin(5\pi x) = F(x) \). Therefore, the solution can also be found by using the formula \( u(x, t) = \frac{1}{2} \left[ F(x - at) + F(x + at) \right] \), where \( a = 3 \). Thus

\[
 u(x, t) = \frac{1}{2} \left[ 4\sin(\pi x + 3t) + 4\sin(\pi x - 3t) - \sin(2\pi x + 3t) + \sin(2\pi x - 3t) - 3\sin(5\pi x + 3t) + 3\sin(5\pi x - 3t) \right]
\]

Indeed, you could easily verify (do this as an exercise) that the solution obtained this way is identical to our previous answer. Just apply the addition formula of sine function \( \sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \) to each term in the above solution and simplify.
**Example:** Solve the one-dimensional wave problem

\[ 9u_{xx} = u_{tt}, \quad 0 < x < 5, \quad t > 0, \]
\[ u(0, t) = 0, \quad u(5, t) = 0, \]
\[ u(x, 0) = 0, \]
\[ ut(x, 0) = 4. \]

As in the previous example, \( a^2 = 9 \) (so \( a = 3 \)), and \( L = 5 \).

Therefore, the general solution remains

\[ u(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos \frac{3n\pi t}{5} + B_n \sin \frac{3n\pi t}{5} \right) \sin \frac{n\pi x}{5}. \]

Now, \( f(x) = 0 \), consequently all \( A_n = 0 \). We just need to find \( B_n \). The initial velocity \( g(x) = 4 \) is a constant function. It is not an odd periodic function. Therefore, we need to expand it into its odd periodic extension (period \( T = 10 \), then equate it with \( u_t(x, 0) \). In short:

\[ B_n = \frac{2}{an\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} dx = \frac{2}{3n\pi} \int_0^5 4 \sin \frac{n\pi x}{5} dx \]

\[ = \begin{cases} 
\frac{80}{3n^2\pi^2}, & n = \text{odd} \\
0, & n = \text{even} 
\end{cases} \]

Therefore,

\[ u(x, t) = \sum_{n=1}^{\infty} \frac{80}{3(2n-1)^2\pi^2} \sin \frac{3(2n-1)\pi t}{5} \sin \frac{(2n-1)\pi x}{5}. \]
The Structure of the Solutions of the Wave Equation

In addition to the fact that the constant $a$ is the standing wave’s propagation speed, several other observations can be readily made from the solution of the wave equation that give insights to the nature of the solution.

To reduce the clutter, let us look at the form of the solution when there is no initial velocity (when $g(x) = 0$). The solution is

$$ u(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{an\pi t}{L} \sin \frac{n\pi x}{L}. $$

The sine terms are functions of $x$. They described the spatial wave patterns (the wavy “shape” of the string that we could visually observe), called the normal modes, or natural modes. The frequencies of those sine waves that we could see, $n\pi/L$, are called the spatial frequencies of the wave. They are also known as the wave numbers. It measures the angular motion, in radians, per unit distance that the wave travels. The “period” of each spatial (sine) function, $2/(n\pi/L) = 2L/n$, is the wave length of each term. Meanwhile, the cosine terms are functions of $t$, they give the vertical displacement of the string relative to its equilibrium position (which is just the horizontal, or the $x$-axis). They describe the up-and-down vibrating motion of the string at each point of the string. These temporal frequencies (the frequencies of functions of $t$; in this case, the cosines’) are the actual frequencies of oscillating motion of vertical displacement. Since this is the undamped wave equation, the motion of the string is simple harmonic. The frequencies of the cosine terms, $an\pi/L$ (measured in radians per second), are called the natural frequencies of the string. In a string instrument, they are the frequencies of the sound that we could hear. The corresponding natural periods ($= 2\pi$/natural frequency) are, therefore, $T = 2L/an$.

For $n = 1$, the observable spatial wave pattern is that of $\sin(\pi x/L)$. The wave length is $2L$, meaning the length $L$ string carries only a half period of the sinusoidal motion. It is the string’s first natural mode. The first natural
frequency of oscillation, \( \frac{a\pi}{L} \), is called the **fundamental frequency** of the string. It is, given the set-up, the lowest frequency note the vibrating string can produce. It is also called, in acoustics, as the **first harmonic** of the string.

For \( n = 2 \), the spatial wave pattern is \( \sin(2\pi x / L) \) is the second natural mode. Its wavelength is \( L \), which is the length of the string itself. The second natural frequency of oscillation, \( 2a\pi / L \), is also called the second harmonic, or the **first overtone**, of the string. It is exactly twice of the string’s fundamental frequency; hence its wavelength (= \( L \)) is only half as long. Acoustically, it produces a tone that is exactly one **octave** higher than the first harmonic. For \( n = 3 \), the third natural frequency, \( 3a\pi / L \), is also called the third harmonic, or the second overtone. It is 3 times larger than the fundamental frequency and, at a 3:2 ratio over the second harmonic, is situated exactly halfway between the adjacent octaves (at the second and the fourth harmonics). The fourth natural frequency (fourth harmonic/ third overtone), \( 4a\pi / L \), is four times larger than the fundamental frequency and twice of that the second natural frequency. The tone it produces is, therefore, exactly 2 octaves and 1 octave higher than those generated by the first and second harmonics, respectively. Together, the sequence of all positive integer multiples of the fundamental frequency is called a **harmonic series** (not to be confused with that **other** harmonic series that you have studied in calculus).

The motion of the string is the combination of all its natural modes, as indicated by the terms of the infinite series of the general solution. The presence, and magnitude, of the nature modes are solely determined by the (Fourier sine series expansion of) initial conditions.

Lastly, notice that the “wavelike” behavior of the solution of the undamped wave equation, quite unlike the solution of the heat conduction equation discussed earlier, does not decrease in amplitude/intensity with time. It never reaches a steady state (unless the solution is trivial, \( u(x,t) = 0 \), which occurs when \( f(x) = g(x) = 0 \)). This is a consequence of the fact that the undamped wave motion is a thermodynamically reversible process that needs not obey the second law of Thermodynamics.
First natural mode (oscillates at the fundamental frequency / 1st harmonic):

Second natural mode (oscillates at the 2nd natural frequency / 2nd harmonic):

Third natural mode (oscillates at the 3rd natural frequency / 3rd harmonic):
Summary of Wave Equation: Vibrating String Problems

The vertical displacement of a vibrating string of length $L$, securely clamped at both ends, of negligible weight and without damping, is described by the homogeneous undamped wave equation initial-boundary value problem:

$$a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0,$$
$$u(0, t) = 0, \text{ and } u(L, t) = 0,$$
$$u(x, 0) = f(x), \text{ and } u_t(x, 0) = g(x).$$

The general solution is

$$u(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos \frac{an\pi t}{L} + B_n \sin \frac{an\pi t}{L} \right) \sin \frac{n\pi x}{L}.$$

The particular solution can be found by the formulas:

$$A_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx,$$
$$B_n = \frac{2}{an\pi} \int_{0}^{L} g(x) \sin \frac{n\pi x}{L} \, dx.$$

The solution waveform has a constant (horizontal) propagation speed, in both directions of the x-axis, of $a$. The vibrating motion has a (vertical) velocity given by $u_t(x, t)$ at any location $0 < x < L$ along the string.
Exercises E-4.1:

1. Solve the vibrating string problem of the given initial conditions.

\[ 4u_{xx} = u_{tt}, \quad 0 < x < \pi, \quad t > 0, \]
\[ u(0, t) = 0, \quad u(\pi, t) = 0, \]

(a) \[ u(x, 0) = 12\sin(2x) - 16\sin(5x) + 24\sin(6x), \]
\[ u_t(x, 0) = 0. \]

(b) \[ u(x, 0) = 0, \]
\[ u_t(x, 0) = 6. \]

(c) \[ u(x, 0) = 0, \]
\[ u_t(x, 0) = 12\sin(2x) - 16\sin(5x) + 24\sin(6x). \]

2. Solve the vibrating string problem.

\[ 100u_{xx} = u_{tt}, \quad 0 < x < 2, \quad t > 0, \]
\[ u(0, t) = 0, \quad \text{and} \quad u(2, t) = 0, \]
\[ u(x, 0) = 32\sin(\pi x) + e^2 \sin(3\pi x) + 25\sin(6\pi x), \]
\[ u_t(x, 0) = 6\sin(2\pi x) - 16\sin(5\pi x / 2). \]

3. Solve the vibrating string problem.

\[ 25u_{xx} = u_{tt}, \quad 0 < x < 1, \quad t > 0, \]
\[ u(0, t) = 0, \quad \text{and} \quad u(2, t) = 0, \]
\[ u(x, 0) = x - x^2, \]
\[ u_t(x, 0) = \pi. \]

4. Verify that the D’Alembert solution, \( u(x, t) = \left[F(x - at) + F(x + at)\right] / 2, \)
where \( F(x) \) is an odd periodic function of period \( 2L \) such that \( F(x) = f(x) \) on the interval \( 0 < x < L, \) indeed satisfies the given initial-boundary value problem by checking that it satisfies the wave equation, boundary conditions, and initial conditions.
\[ a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0, \]
\[ u(0, t) = 0, \quad u(L, t) = 0, \]
\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \]

5. Use the method of separation of variables to solve the following wave equation problem where the string is rigid, but not fixed in place, at both ends (i.e., it is inflexible at the endpoints such that the slope of displacement curve is always zero at both ends, but the two ends of the string are allowed to freely slide in the vertical direction).

\[ a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0, \]
\[ u_x(0, t) = 0, \quad u_x(L, t) = 0, \]
\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \]

6. What is the steady-state displacement of the string in #5? What is \( \lim_{t \to \infty} u(x, t) \)? Are they the same?
Answers E-4.1:

1. (a) \( u(x, t) = 12\cos(4t) \sin(2x) - 16\cos(10t) \sin(5x) + 24\cos(12t) \sin(6x). \)
   (c) \( u(x, t) = 3\sin(4t) \sin(2x) - 1.6\sin(10t) \sin(5x) + 2\sin(12t) \sin(6x). \)

5. The general solution is
   
   \[
   u(x, t) = A_0 + B_0 \cdot t + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi t}{L} + B_n \sin \frac{n\pi t}{L} \right) \cos \frac{n\pi x}{L}.
   \]

   The particular solution can be found by the formulas:
   
   \[
   A_0 = \frac{1}{L} \int_0^L f(x) \, dx, \quad A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx, \quad B_0 = \frac{1}{L} \int_0^L g(x) \, dx, \quad \text{and}
   \]
   
   \[
   B_n = \frac{2}{an\pi} \int_0^L g(x) \cos \frac{n\pi x}{L} \, dx.
   \]

6. The steady-state displacement is the constant term of the solution, \( A_0. \)
   The limit does not exist unless \( u(x, t) = C \) is a constant function, which happens when \( f(x) = C \) and \( g(x) = 0, \) in which case the limit is \( C. \) They are not the same otherwise.
The General Wave Equation

The most general form of the one-dimensional wave equation is:

$$a^2 u_{xx} + F(x,t) = u_{tt} + \gamma u_t + ku.$$  

Where

- $a$ = the propagation speed of the wave,
- $\gamma$ = the damping constant
- $k$ = (external) restoration factor, such as when vibrations occur in an elastic medium.
- $F(x, t)$ = arbitrary external forcing function (If $F = 0$ then the equation is homogeneous, else it is nonhomogeneous.)
The Telegraph Equation

The most well-known example of (a homogeneous version of) the general wave equation is the telegraph equation. It describes the voltage $u(x, t)$ inside a piece of telegraph / transmission wire, whose electrical properties per unit length are: resistance $R$, inductance $L$, capacitance $C$, and conductance of leakage current $G$:

$$a^2 u_{xx} = u_{tt} + \gamma u_t + ku.$$

Where $a^2 = 1/LC$, $\gamma = G/C + R/L$, and $k = GR/CL$. 
Example: The One-Dimensional Damped Wave Equation

\[ a^2 u_{xx} = u_{tt} + \gamma u_t, \quad \gamma \neq 0. \]

Suppose boundary conditions remain as the same (both ends fixed): \((0, t) = 0,\) and \(u(L, t) = 0.\)

The equation can be separated as follow. First rewrite it as:

\[ a^2 X'' T = X T'' + \gamma X T', \]

Divide both sides by \(a^2 X T,\) and insert a constant of separation:

\[ \frac{X''}{X} = \frac{T'' + \gamma T'}{a^2 T} = -\lambda. \]

Rewrite it into 2 equations:

\[ X'' = -\lambda X \quad \rightarrow \quad X'' + \lambda X = 0, \]

\[ T'' + \gamma T' = -a^2 \lambda T \quad \rightarrow \quad T'' + \gamma T' + a^2 \lambda T = 0. \]

The boundary conditions also are separated, as usual:

\[ u(0, t) = 0 \quad \rightarrow \quad X(0)T(t) = 0 \quad \rightarrow \quad X(0) = 0 \quad \text{or} \quad T(t) = 0 \]

\[ u(L, t) = 0 \quad \rightarrow \quad X(L)T(t) = 0 \quad \rightarrow \quad X(L) = 0 \quad \text{or} \quad T(t) = 0 \]

As before, setting \(T(t) = 0\) would result in the constant zero solution only. Therefore, we must choose the two (nontrivial) conditions in terms of \(x:\)

\[ X(0) = 0, \quad \text{and} \quad X(L) = 0. \]
After separation of variables, we have the system

\[ X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(L) = 0, \]

\[ T'' + \gamma T' + \alpha^2 \lambda^2 T = 0. \]

The next step is to find the eigenvalues and their corresponding eigenfunctions of the boundary value problem

\[ X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(L) = 0. \]

This is a familiar problem that we have encountered more than once previously. The eigenvalues and eigenfunctions are, recall,

**Eigenvalues:** \( \lambda = \frac{n^2 \pi^2}{L^2}, \quad n = 1, 2, 3, \ldots \)

**Eigenfunctions:** \( X_n = \sin \frac{n \pi x}{L}, \quad n = 1, 2, 3, \ldots \)

The equation of \( t \), however, has different kind of solutions depending on the roots of its characteristic equation.
(Optional topic) Nonhomogeneous Undamped Wave Equation

Problems of partial differential equation that contains a nonzero forcing function (which would make the equation itself a nonhomogeneous partial differential equation) can sometimes be solved using the same idea that we have used to handle nonhomogeneous boundary conditions – by considering the solution in 2 parts, a steady-state part and a transient part. This is possible when the forcing function is independent of time \( t \), which then could be used to determine the steady-state solution. The transient solution would then satisfy a certain homogeneous equation. The 2 parts are thus solved separately and their solutions are added together to give the final result. Let us illustrate this idea with a simple example: when the string’s weight is no longer “negligible”.

Example: A flexible string of length \( L \) has its two ends firmly secured. Assume there is no damping. Suppose the string has a weight density of 1 Newton per meter. That is, it is subject to, uniformly across its length, a constant force of \( F(x, t) = 1 \) unit per unit length due to its own weight. Let \( u(x, t) \) be the vertical displacement of the string, \( 0 < x < L \), and at any time \( t > 0 \). It satisfies the nonhomogeneous one-dimensional undamped wave equation:

\[
a^2 u_{xx} + 1 = u_t.
\]

The usual boundary conditions \( u(0, t) = 0 \), and \( u(L, t) = 0 \), apply. Plus the initial conditions \( u(x, 0) = f(x) \) and \( u_t(x, 0) = g(x) \).

Since the forcing function is independent of time \( t \), its effect is to impart, permanently, a displacement on the string that depends only on the location (the effect is subject to the boundary conditions, thus might change with \( x \)). That is, the effect is to introduce a nonzero
steady-state displacement, \( v(x) \). Hence, we can rewrite the solution \( u(x,t) \) as:

\[
 u(x,t) = v(x) + w(x,t).
\]

By setting \( t \) to be a constant and rewrite the equation and the boundary conditions to be dependent of \( x \) only, the steady-state solution \( v(x) \) must satisfy:

\[
 a^2 v'' + 1 = 0, \\
v(0) = 0, \quad v(L) = 0.
\]

Rewrite the equation as \( v'' = -1/a^2 \), and integrate twice, we get

\[
 v(x) = \frac{-1}{2a^2} x^2 + C_1 x + C_2.
\]

Apply the boundary conditions to find \( C_1 = L/2a^2 \) and \( C_2 = 0 \):

\[
 v(x) = \frac{-1}{2a^2} x^2 + \frac{L}{2a^2} x.
\]

Comment: Thus, the sag of a wire or cable due to its own weight can be seen as a manifestation of the steady-solution of the wave equation. The sag is also parabolic, rather than sinusoidal, as one might have reasonably assumed, in nature.

We can then subtract out \( v(x) \) from the equation, boundary conditions, and the initial conditions (try this as an exercise), the transient solution \( w(x,t) \) must satisfy:

\[
 a^2 w_{xx} = w_{tt}, \quad 0 < x < L, \quad t > 0, \\
w(0,t) = 0, \quad w(L,t) = 0, \\
w(x,0) = f(x) - v(x), \quad w_t(x,0) = g(x).
\]
The problem is now transformed to the homogeneous problem we have already solved. The solution is just

$$w(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos \frac{an \pi t}{L} + B_n \sin \frac{an \pi t}{L} \right) \sin \frac{n \pi x}{L}.$$  

Combining the steady-state and transient solutions, the general solution is found to be

$$u(x, t) = v(x) + w(x, t) = -1 \frac{1}{2a^2} x^2 + \frac{L}{2a^2} x + \sum_{n=1}^{\infty} \left( A_n \cos \frac{an \pi t}{L} + B_n \sin \frac{an \pi t}{L} \right) \sin \frac{n \pi x}{L}.$$  

The coefficients can be calculated and the particular solution determined by using the formulas:

$$A_n = \frac{2}{L} \int_{0}^{L} (f(x) - v(x)) \sin \frac{n \pi x}{L} \, dx,$$

and

$$B_n = \frac{2}{an \pi} \int_{0}^{L} g(x) \sin \frac{n \pi x}{L} \, dx.$$  

Note: Since the velocity $u_t(x, t) = v_t(x) + w_t(x, t) = 0 + w_t(x, t) = w_t(x, t)$. The initial velocity does not need any adjustment, as $u_t(x, 0) = w_t(x, 0) = g(x)$.

Comment: We can clearly see that, even though a nonzero steady-state solution exists, the displacement of the string still will not converge to it as $t \to \infty$.
The Laplace Equation / Potential Equation

The last type of the second order linear partial differential equation in 2 independent variables is the two-dimensional Laplace equation, also called the potential equation. Unlike the other equations we have seen, a solution of the Laplace equation is always a steady-state (i.e. time-independent) solution. Indeed, the variable \( t \) is not even present in the Laplace equation. The Laplace equation describes systems that are in a state of equilibrium whose behavior does not change with time. Some applications of the Laplace equation are finding the potential function of an object acted upon by a gravitational / electric / magnetic field, finding the steady-state temperature distribution of the (2- or 3-dimensional) heat conduction equation, and the steady-state flow of an ideal fluid (where the flow velocity forms a vector field that has zero curl and zero divergence).

Since the time variable is not present in the Laplace equation, any problem of the Laplace equation will not, therefore, have any initial condition. A Laplace equation problem has only boundary conditions.

Let \( u(x, y) \) be the potential function at a point \((x, y)\), then it is governed by the two-dimensional Laplace equation

\[
 u_{xx} + u_{yy} = 0.
\]

Any real-valued function having continuous first and second partial derivatives that satisfies the two-dimensional Laplace equation is called a harmonic function.

Similarly, suppose \( u(x, y, z) \) is the potential function at a point \((x, y, z)\), then it is governed by the three-dimensional Laplace equation

\[
 u_{xx} + u_{yy} + u_{zz} = 0.
\]
**Comment:** The one-dimensional Laplace equation is rather dull. It is merely \( u_{xx} = 0 \), where \( u \) is a function of \( x \) alone. It is not a partial differential equation, but rather a simple integration problem of \( u'' = 0 \). (What is its solution? Where have we seen it just very recently?)

The boundary conditions that accompany a 2-dimensional Laplace equation describe the conditions on the boundary curve that encloses the 2-dimensional region in question. While those accompany a 3-dimensional Laplace equation describe the conditions on the boundary surface that encloses the 3-dimensional spatial region in question.
The Relationships among Laplace, Heat, and Wave Equations
(Optional topic)

Now let us take a step back and see the bigger picture: how the homogeneous heat conduction and wave equations are structured, and how they are related to the Laplace equation of the same spatial dimension.

Suppose $u(x,y)$ is a function of two variables, the expression $u_{xx} + u_{yy}$ is called the Laplacian of $u$. It is often denoted by

$$\nabla^2 u = u_{xx} + u_{yy}.$$  

Similarly, for a three-variable function $u(x,y,z)$, the 3-dimensional Laplacian is then

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz}.$$  

(As we have just noted, in the one-variable case, the Laplacian of $u(x)$, degenerates into $\nabla^2 u = u''$.)

The homogeneous heat conduction equations of 1-, 2-, and 3- spatial dimension can then be expressed in terms of the Laplacians as:

$$\alpha^2 \nabla^2 u = u_t,$$

where $\alpha^2$ is the thermo diffusivity constant of the conducting material. Thus, the homogeneous heat conduction equations of 1-, 2-, and 3- dimension are, respectively,

$$\alpha^2 u_{xx} = u_t$$

$$\alpha^2 (u_{xx} + u_{yy}) = u_t$$

$$\alpha^2 (u_{xx} + u_{yy} + u_{zz}) = u_t$$
As well, the homogeneous wave equations of 1-, 2-, and 3- spatial dimension can then be similarly expressed in terms of the Laplacians as:

\[ a^2 \nabla^2 u = u_{tt}, \]

where the constant \( a \) is the propagation velocity of the wave motion. Thus, the homogeneous wave equations of 1-, 2-, and 3-dimension are, respectively,

\[ a^2 u_{xx} = u_{tt}, \]
\[ a^2 (u_{xx} + u_{yy}) = u_{tt}, \]
\[ a^2 (u_{xx} + u_{yy} + u_{zz}) = u_{tt}. \]

Now let us consider the steady-state solutions of these heat conduction and wave equations. In each case, the steady-state solution, being independent of time, must have all zero as its partial derivatives with respect to \( t \). Therefore, in every instance, the steady-state solution can be found by setting, respectively, \( u_t \) or \( u_{tt} \) to zero in the heat conduction or the wave equations and solve the resulting equation. That is, the steady-state solution of a heat conduction equation satisfies

\[ a^2 \nabla^2 u = 0, \]

and the steady-state solution of a wave equation satisfies

\[ a^2 \nabla^2 u = 0. \]

\(^\dagger\) Even the electromagnetic waves are described by this equation. It can be easily shown by vector calculus that any electric field \( E \) and magnetic field \( B \) satisfying the Maxwell’s Equations will also satisfy the 3-dimensional wave equation, with propagation speed \( a = c \approx 299792 \text{ km/s} \), the speed of light in vacuum.
In all cases, we can divide out the (always positive) coefficient $\alpha^2$ or $a^2$ from the equations, and obtain a “universal” equation:

\[ \nabla^2 u = 0. \]

This universal equation that all the steady-state solutions of heat conduction and wave equations have to satisfy is the Laplace/potential equation!

Consequently, the 1-, 2-, and 3-dimensional Laplace equations are, respectively,

\[ u_{xx} = 0, \]
\[ u_{xx} + u_{yy} = 0, \]
\[ u_{xx} + u_{yy} + u_{zz} = 0. \]

Therefore, the Laplace equation, among other applications, is used to solve the steady-state solution of the other two types of equations. And all solutions of a Laplace equation are steady-state solutions. To answer the earlier question, we have had seen and used the one-dimensional Laplace equation (which, with only one independent variable, $x$, is a very simple ordinary differential equation, $u'' = 0$, and is not a PDE) when we were trying to find the steady-state solution of the one-dimensional homogeneous heat conduction equation earlier.
Laplace Equation for a rectangular region

Consider a rectangular region of length $a$ and width $b$. Suppose the top, bottom, and left sides border free-space; while beyond the right side there lies a source of heat/gravity/magnetic flux, whose strength is given by $f(y)$. The potential function at any point $(x, y)$ within this rectangular region, $u(x, y)$, is then described by the boundary value problem:

\[
\begin{align*}
(\text{2-dim. Laplace eq.}) & \quad u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b, \\
(\text{Boundary conditions}) & \quad u(x, 0) = 0, \quad u(x, b) = 0, \\
& \quad u(0, y) = 0, \quad u(a, y) = f(y).
\end{align*}
\]

The separation of variables proceeds similarly. A slight difference here is that $Y(y)$ is used in the place of $T(t)$. Let $u(x, y) = X(x)Y(y)$ and substituting $u_{xx} = X''Y$ and $u_{yy} = XY''$ into the wave equation, it becomes

\[
X''Y + XY''' = 0,
\]

\[
X''Y = -XY''.
\]

Dividing both sides by $XY$:

\[
\frac{X''}{X} = -\frac{Y''}{Y}
\]

Now that the independent variables are separated to the two sides, we can insert the constant of separation. Unlike the previous instances, it is more convenient to denote the constant as positive $\lambda$ instead.

\[
\frac{X''}{X} = -\frac{Y''}{Y} = \lambda.
\]
\[
\frac{X''}{X} = \lambda \quad \rightarrow \quad X'' = \lambda X \quad \rightarrow \quad X'' - \lambda X = 0,
\]
\[
\frac{-Y''}{Y} = \lambda \quad \rightarrow \quad Y'' = -\lambda Y \quad \rightarrow \quad Y'' + \lambda Y = 0.
\]

The boundary conditions also separate:

\[
u(x, 0) = 0 \rightarrow X(x) = 0 \quad \text{or} \quad Y(0) = 0
\]
\[
u(x, b) = 0 \rightarrow X(x) = 0 \quad \text{or} \quad Y(b) = 0
\]
\[
u(0, y) = 0 \rightarrow X(0) = 0 \quad \text{or} \quad Y(y) = 0
\]
\[
u(a, y) = f(y) \rightarrow X(a) = f(y) \quad \text{[cannot be simplified further]}
\]

As usual, in order to obtain nontrivial solutions, we need to ignore the constant zero function in the solution sets above, and instead choose \(Y(0) = 0, \ Y(b) = 0, \) and \(X(0) = 0\) as the new boundary conditions. The fourth boundary condition, however, cannot be simplified this way. So we shall leave it as-is. (Don’t worry. It will play a useful role later.) The result, after separation of variables, is the following simultaneous system of ordinary differential equations, with a set of boundary conditions:

\[
\begin{align*}
X'' - \lambda X &= 0, \quad X(0) = 0, \\
Y'' + \lambda Y &= 0, \quad Y(0) = 0 \quad \text{and} \quad Y(b) = 0.
\end{align*}
\]

Plus the fourth boundary condition, \(u(a, y) = f(y).\)

The next step is to solve the eigenvalue problem. Notice that there is another slight difference. Namely that this time it is the equation of \(Y\) that gives rise to the two-point boundary value problem which we need to solve.
\[ Y'' + \lambda Y = 0, \quad Y(0) = 0, \quad Y(b) = 0. \]

However, except for the fact that the variable is \( y \) and the function is \( Y \), rather than \( x \) and \( X \), respectively, we have already seen this problem before (more than once, as a matter of fact; here the constant \( L = b \)). The eigenvalues of this problem are

\[ \lambda = \sigma^2 = \frac{n^2 \pi^2}{b^2}, \quad n = 1, 2, 3, \ldots \]

Their corresponding eigenfunctions are

\[ Y_n = \sin \left( \frac{n \pi y}{b} \right), \quad n = 1, 2, 3, \ldots \]

Once we have found the eigenvalues, substitute \( \lambda \) into the equation of \( x \). We have the equation, together with one boundary condition:

\[ X'' - \frac{n^2 \pi^2}{b^2} X = 0, \quad X(0) = 0. \]

Its characteristic equation, \( r^2 - \frac{n^2 \pi^2}{b^2} = 0 \), has real roots \( r = \pm \frac{n \pi}{b} \).

Hence, the general solution for the equation of \( x \) is

\[ X = C_1 e^{\frac{n \pi x}{b}} + C_2 e^{-\frac{n \pi x}{b}}. \]

The single boundary condition gives

\[ X(0) = 0 = C_1 + C_2 \quad \rightarrow \quad C_2 = -C_1. \]
Therefore, for \( n = 1, 2, 3, \ldots \),

\[
X_n = C_n \left( e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}} \right).
\]

Because of the identity for the hyperbolic sine function

\[
\sinh \theta = \frac{e^\theta - e^{-\theta}}{2},
\]

the previous expression is often rewritten in terms of hyperbolic sine:

\[
X_n = K_n \sinh \frac{n\pi x}{b}, \quad n = 1, 2, 3, \ldots
\]

The coefficients satisfy the relation:

\[
K_n = 2C_n.
\]

Combining the solutions of the two equations, we get the set of solutions that satisfies the two-dimensional Laplace equation, given the specified boundary conditions:

\[
u_n(x, y) = X_n(x) Y_n(y) = K_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}, \quad n = 1, 2, 3, \ldots \]

The general solution, as usual, is just the linear combination of all the above, linearly independent, functions \( u_n(x, y) \). That is,

\[
u(x, y) = \sum_{n=1}^{\infty} K_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}.
\]
This solution, of course, is specific to the set of boundary conditions
\[ u(x, 0) = 0, \quad u(x, b) = 0, \]
\[ u(0, y) = 0, \quad u(a, y) = f(y). \]

To find the particular solution, we will use the fourth boundary condition, namely, \( u(a, y) = f(y) \).

\[
u(a, y) = \sum_{n=1}^{\infty} K_n \sinh \frac{a n \pi}{b} \sin \frac{n \pi y}{b} = f(y)
\]

We have seen this story before. There is nothing really new here. The summation above is a sine series whose Fourier sine coefficients are \( b_n = K_n \sinh(an \pi / b) \). Therefore, the above relation says that the last boundary condition, \( f(y) \), must either be an odd periodic function (period = \( 2b \)), or it needs to be expanded into one. Once we have \( f(y) \) as a Fourier sine series, the coefficients \( K_n \) of the particular solution can then be computed:

\[
K_n \sinh \frac{a n \pi}{b} = b_n = \frac{2}{b} \int_{0}^{b} f(y) \sin \frac{n \pi y}{b} \, dy
\]

Therefore,

\[
K_n = \frac{b_n}{\sinh \frac{a n \pi}{b}} = \frac{2}{b \sinh \frac{a n \pi}{b}} \int_{0}^{b} f(y) \sin \frac{n \pi y}{b} \, dy.
\]
(Optional topic) Laplace Equation in Polar Coordinates

The steady-state solution of the two-dimensional heat conduction or wave equation within a circular region (the interior of a circular disc of radius \( k \), that is, on the region \( r < k \)) in polar coordinates, \( u(r, \theta) \), is described by the polar version of the two-dimensional Laplace equation

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.
\]

The boundary condition, in this set-up, specifying the condition on the circular boundary of the disc, i.e., on the curve \( r = k \), is given in the form \( u(k, \theta) = f(\theta) \), where \( f \) is a function defined on the interval \([0, 2\pi)\). Note that there is only one set of boundary condition, prescribed on a circle. This will cause a slight complication. Furthermore, the nature of the coordinate system implies that \( u \) and \( f \) must be periodic functions of \( \theta \), of period \( 2\pi \). Namely, \( u(r, \theta) = u(r, \theta + 2\pi) \), and \( f(\theta) = f(\theta + 2\pi) \).

By letting \( u(r, \theta) = R(r)\Theta(\theta) \), the equation becomes

\[
R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0.
\]

Which can then be separated to obtain

\[
\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.
\]

This equation above can be rewritten into two ordinary differential equations:

\[
r^2 R'' + rR' - \lambda R = 0,
\]

\[
\Theta'' + \lambda \Theta = 0.
\]
The eigenvalues are not found by straightforward computation. Rather, they are found by a little deductive reasoning. Based solely on the fact that $\Theta$ must be a periodic function of period $2\pi$, we can conclude that $\lambda = 0$ and $\lambda = n^2, n = 1, 2, 3\ldots$, are the eigenvalues. The corresponding eigenfunctions are $\Theta_0 = 1$ and $\Theta_n = A_n \cos n\theta + B_n \sin n\theta$. The equation of $r$ is an Euler equation (the solution of which is outside of the scope of this course).

The general solution of the Laplace equation in polar coordinates is

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left( A_n \cos n\theta + B_n \sin n\theta \right) r^n.$$  

Applying the boundary condition $u(k, \theta) = f(\theta)$, we see that

$$u(k, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left( A_n k^n \cos n\theta + B_n k^n \sin n\theta \right) = f(\theta).$$

Since $f(\theta)$ is a periodic function of period $2\pi$, it would already have a suitable Fourier series representation. Namely,

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos n\theta + b_n \sin n\theta \right).$$

Hence, $A_0 = a_0$, $A_n = a_n / k^n$, and $B_n = b_n / k^n$, $n = 1, 2, 3\ldots$

For a problem on the unit circle, whose radius $k = 1$, the coefficients $A_n$ and $B_n$ are exactly identical to, respectively, the Fourier coefficients $a_n$ and $b_n$ of the boundary condition $f(\theta)$. 

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(Optional topic) Undamped Wave Equation in Polar Coordinates

The vibrating motion of an elastic membrane that is circular in shape can be described by the two-dimensional wave equation in polar coordinates:

\[ u_{rr} + \left( \frac{1}{r} \right) u_r + \left( \frac{1}{r^2} \right) u_{\theta\theta} = a^{-2} u_{tt}. \]

The solution is \( u(r, \theta, t) \), a function of 3 independent variables that describes the vertical displacement of each point \( (r, \theta) \) of the membrane at any time \( t \).