Higher Derivatives

Since the derivative $f'$ of a differentiable function $f$ is itself a function, it can be differentiated again. The result (which, if exists, is again a function itself) is called the second derivative of $f$, denoted by $f''$. (The second derivative represents the rate of change of the rate of change of $f$.)

*Comment:* Think the notation as $(f'(x))' = f''(x)$.

Alternative notations for the second derivative:

\[
 f''(x) = y'' = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2 y}{dx^2} = \frac{d^2 f}{dx^2} = \frac{d^2}{dx^2} f(x) = D^2 f(x)
\]

ex. Suppose $f(x) = x^2 \sin x$. Find $f'(x)$ and $f''(x)$.

\[
 f'(x) = x^2 (\sin x)' + (x^2)' \sin x = x^2 \cos x + 2x \sin x
\]

\[
 f''(x) = (x^2 \cos x)' + (2x \sin x)' = [x^2 (\cos x)'+ (x^2)' \cos x] + \\
 2[x(\sin x)'+ (x)' \sin x] = [x^2(-\sin x) + 2x \cos x] + 2[x \cos x + \sin x] \\
 = -x^2 \sin x + 4x \cos x + 2 \sin x
\]

Since the second derivative is again a function, it can have a derivative of its own. The derivative of the second derivative of $f$ is the third derivative of $f$, denoted by $f'''$. This process can be repeated to find the fourth, fifth, sixth, … derivatives, as long as each successive derivative exists. The “prime” notation becomes too cumbersome to write past the third derivative, so the $n$-th derivative of $f$, for $n$ greater than 3, is usually written as $f^{(n)}$

Notations for the $n$-th derivative:

\[
 f^{(n)}(x) = y^{(n)} = \frac{d^n y}{dx^n} = \frac{d^n f}{dx^n} = \frac{d^n}{dx^n} f(x) = D^n f(x)
\]
ex. Let $s(t)$ be an object’s displacement/position function. Then $s'(t)$ is the velocity function, $s''(t)$ is its acceleration, and $s'''(t)$ is its jerk.

Comment: In this case, when the independent variable is $t$ instead of $x$, the notations should be changed accordingly, for example:

$$s^{(n)}(t) = \frac{d^n s}{dt^n} = \frac{d^n}{dt^n}s(t)$$

ex. $s(t) = t^2 + 7t - 6$

$$v(t) = s'(t) = 2t + 7$$
$$a(t) = s''(t) = 2$$
$$j(t) = s'''(t) = 0$$

ex. Let $f(x) = x^n$

By repeated uses of the power rule:

$$f'(x) = nx^{n-1}$$
$$f''(x) = n(n-1)x^{n-2}$$
$$f'''(x) = n(n-1)(n-2)x^{n-3}$$
$$f^{(4)}(x) = n(n-1)(n-2)(n-3)x^{n-4}$$
$$\vdots$$
$$f^{(n)}(x) = n(n-1)(n-2)\ldots(3)(2)(1)x^{n-n} = (n!)x^0 = n!$$
$$f^{(n+1)}(x) = 0$$
Higher Derivatives of $\sin x$ and $\cos x$

Let $f(x) = \sin x$ and $g(x) = \cos x$, then

\[
\begin{align*}
 f(x) &= \sin x & g(x) &= \cos x \\
 f'(x) &= \cos x & g'(x) &= -\sin x \\
 f''(x) &= -\sin x & g''(x) &= -\cos x \\
 f'''(x) &= -\cos x & g'''(x) &= \sin x \\
 f^{(4)}(x) &= \sin x & g^{(4)}(x) &= \cos x & \text{<= original functions} \\
 f^{(5)}(x) &= \cos x & g^{(5)}(x) &= -\sin x \\
 f^{(6)}(x) &= -\sin x & g^{(6)}(x) &= -\cos x \\
 f^{(7)}(x) &= -\cos x & g^{(7)}(x) &= \sin x \\
 f^{(8)}(x) &= \sin x & g^{(8)}(x) &= \cos x & \text{<= original functions}
\end{align*}
\]

That is, the derivatives of $\sin x$ and $\cos x$ repeat every 4 times you differentiate them. (Such behavior is similar to the various powers of $i$, the square root of $-1$: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc.)

ex. Find the 1027\textsuperscript{th} derivative of $g(x) = \cos x$.

Since 1027 has a remainder of 3 when divided by 4, the 1027\textsuperscript{th} derivative is the same as the 3\textsuperscript{rd} derivative, therefore,

\[ g^{(1027)}(x) = g'''(x) = \sin x \]