Problem Set 13: Lyapunov Functions and a Relaxation Oscillator

1. Show $V(S,I) = S - S^*(1 + \ln \frac{S}{S^*}) + I - I^*(1 + \ln \frac{I}{I^*})$ is a Lyapunov Function of the nonlinear autonomous mass-action system of differential equations

$$\frac{dS}{dt} = m - \beta SI - dS,$$
$$\frac{dI}{dt} = \beta SI - fI$$

when (S^*, I^*) is a positive stationary solution.

Answer:
The stationary solution $(S^*, I^*) = (f/\beta, (m\beta - df)/(f\beta))$.

$$\begin{aligned}\dot{V} &= \dot{S} - \frac{S^*}{S}\dot{S} + \dot{I} - \frac{I^*}{I}\dot{I} \\ &= m - dS - fI - \frac{S^*}{S}(m - \beta IS - dS) - I^*(\beta S - f) \\ &= -\frac{m(\beta S - f)^2}{\beta f S}\end{aligned}$$

So V is a weak Lyapunov function with minimum $V(S^*, I^*) = 0$. With a little more work, we can show global stability (from positive initial conditions) as long as $I^* > 0$.

2. Use the theory of Lyapunov functions and the energy function

$$V(\theta, \eta) = \frac{1}{2}\eta^2 + 1 - \cos\theta$$

to study the stability of

$$\begin{aligned}\dot{\theta} &= \eta \\ \dot{\eta} &= -\sin(\theta) - f(\theta,\eta)\eta\end{aligned}$$

under the conditions $f(\theta,\eta) > 0, f(\theta,\eta) = f(-\theta,-\eta), f(\theta,\eta) = f(\theta + 2\pi, \eta)$.

Answer:
First, observe that $(\theta, \eta) = (0,0)$ is a stationary solution. Now,

$$\begin{aligned}\dot{V} &= \eta\dot{\eta} + (\sin\theta)\dot{\theta} \\ &= \eta(-\sin\theta - f(\theta,\eta)\eta) + \eta\sin\theta \\ &= -\eta^2 f(\theta,\eta)\end{aligned}$$

So $\dot{V} \leq 0$, making V a weak Lyapunov function. It is not a strong Lyapunov function because $\dot{V}(\theta, 0) = 0$ for all θ, not just $\theta = 0$. In particular, every point $(x, 0)$ with $x \in \{\pi z : z \in \mathbb{Z}\}$ is a stationary solution, and hence an ω-limit set of some non-empty set of initial conditions. So there is no global attractor in this system, but this set of stationary solutions identifies all local attractors. The stationary points where $V(\theta, \eta) = 2$ must be locally unstable, because small perturbations to θ lead to $V < 2$, and V is never increasing. The points where $V(\theta, \eta) = 0$ are locally stable, since the is a local (and global) minimum of V. Every trajectory starting in the set $S = \{(\theta, 0) : \theta \notin \pi\mathbb{Z}\}$ leave that set in short time because $\eta = 0, \sin\theta \neq 0 \to \dot{\eta} \neq 0$. So the stationary points must be the only possible ω-limits. It follows that points where $V(\theta, \eta) = 0$ are locally asymptotically stable. V is smooth and has no saddles for $V < 2$, so the basins of attraction of locally-stable stationary solution include the surrounding compact sets $\{(x, y) : V(x, y) < 2\}$ (The basin's of attraction are actually larger, but to determine these, we have determine the shapes of the stable manifolds of the saddle-points, which depend on f). While we have imposed physical constraints on the periodicity and symmetry of f, only the positivity condition was necessary for these stability results. Thus, any positive frictional force depending on the angle and angular velocity will result in dynamics that roughly agree with our intuition about Galileon pendulums.

3. Find a relaxation-oscillator solution of the autonomous nonlinear system of differential equations

$$\frac{dx}{dt} = \lambda \left[\frac{a(x+q)}{x-q} + x(1-x) - y \right]$$
$$\frac{dy}{dt} = x - y$$

when $0 < q \ll 1$ and $a = 1/20$ in the limit as $\lambda \to \infty$. Approximate the period of the solution.