Problem Set 12: Method of Averaging

1. Use the methods from class that we applied to the van Der Pool equation to determine the dynamics of the Duffing equation

$$\ddot{x} = -x + \epsilon x^3$$

when $|\epsilon| << 1$.

Answer:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 + \epsilon x^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ sint & cost \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \Bigg[\begin{matrix} -\sin(t)\,\epsilon\left(\cos^3(t)\,u^3 + 3\cos(t)\,uv^2 - 3u\cos^3(t)\,v^2 + 3\cos^2(t)\,u^2\sin(t)\,v - \sin(t)\,v^3\,(\text{co} \\ \left((\cos(t))^3 u^3 + 3\cos(t)\,uv^2 - 3u\,(\cos(t))^3 v^2 + 3\,(\cos(t))^2 u^2\sin(t)\,v - \sin(t)\,v^3\,(\cos(t))^2\right. \end{matrix}$$

2. Use the method of averaging to determine the stability of the damped Mathieu equation

$$\ddot{x} = -\gamma\dot{x} - (1 - q\cos\omega t)x$$

for large ω.

Answer:
Using the change of variables $t = s/\omega$, $dx/ds = dx/dt/\omega$, $d^2x/ds^2 = d^2x/dt^2/\omega^2$. Thus,

$$\frac{d^2x}{ds^2} = \frac{1}{\omega^2}\left\{-\gamma\omega\frac{dx}{ds} - (1 - q\cos s)x\right\} = \frac{1}{\omega}\left\{-\gamma\frac{dx}{ds} - \frac{(1 - q\cos s)}{\omega}x\right\}$$

This form satisfies the assumptions of the averaging theorem for large ω, so we can now approximate the solutions using the average dynamics when $1/\omega \to 0$. In this case, the average is trivial, since $(1 - q\cos s)/\omega \to 0$ as $\omega \to \infty$. The averaged equation is then

$$\frac{d^2\tilde{x}}{ds^2} = -\frac{\gamma}{\omega}\frac{d\tilde{x}}{ds}$$

This equation has solutions $\tilde{x}(s) = C_0 + C_1 e^{-\gamma s/\omega}$, so $x(t) \approx C_0 + C_1 e^{-\gamma t}$. This is an interesting solution because it suggests any constant solution can be almost stable under sufficiently fast oscillations.