Math 511, Autumn 2010

Problem Set 10: Boundary Layers, Dimensional Analysis

1. In many physical problems, the behavior of systems near their boundaries are different from their behavior far from their boundaries. When this happens, the solution near the boundary is often called a boundary-layer. An elementary example of this the given by

$$\epsilon \ddot{x} + \dot{x} - x = 0, \quad x(0) = x(1) = 1, \quad 0 < \epsilon \ll 1.$$

   (a) Find an exact solution to this linear homogeneous second-order equation.
   Answer:

$$x(t) = \frac{(1-e^b)e^{at}}{e^a - e^b} + \frac{(e^a - 1)e^{bt}}{e^a - e^b}$$

   where $a = \frac{-1+\sqrt{1-4\epsilon}}{2\epsilon}$ and $b = \frac{-1-\sqrt{1-4\epsilon}}{2\epsilon}$.

   (b) Find a solution in the case of $\epsilon = 0$, and explain why this can not be used as a uniform approximation to the full solution. (This is called an "outer" solution, $x_{outer}(t)$)
   Answer:
   Taking $\epsilon = 0$, $\dot{x}_{outer}(t) - x_{outer}(t) = 0$ so $x_{outer}(t) = C_o e^t$. If $x_{outer}(0) = 1$, then $x_{outer}(1) = e^1 \neq 1$. If $x_{outer}(1) = 1$, then $x_{outer}(0) = e^{-1} \neq 1$. So $x_{outer}(t)$ can not match both boundary conditions.

   (c) Use the change of variables $y(s) = x(\epsilon s)$ to derive an equation for $y(s)$ that is second order, even when $\epsilon = 0$.
   Answer:
   Under this change of variables,

$$\frac{\partial y}{\partial s} = \frac{\partial x}{\partial t}\frac{\partial t}{\partial s} = \frac{\partial x}{\partial t}\epsilon, \quad \frac{\partial^2 y}{\partial s^2} = \frac{\partial^2 x}{\partial t^2}\epsilon^2.$$

   Substituting into the original equation,

$$\epsilon \frac{1}{\epsilon^2}\frac{\partial^2 y}{\partial s^2} + \frac{1}{\epsilon}\frac{\partial y}{\partial s} - y(s) = 0$$

   After multiplying by ϵ, we find

$$\frac{\partial^2 y}{\partial s^2} + \frac{\partial y}{\partial s} - \epsilon y = 0$$

(d) Solve this equation when $\epsilon = 0$ to determine $x_{inner}(t) = y(t/\epsilon)$.

Answer:

When $\epsilon = 0$, $y(s) = C_1 + C_2 e^{-s}$ leaving $x_{inner}(t) = C_1 + C_2 e^{-t/\epsilon}$.

(e) Use the boundary conditions and the matching condition $x_{inner}(\epsilon) = x_{outer}(\epsilon)$ to construct a piecewise-smooth uniform approximation of exact solution $x(t)$.

Answer:

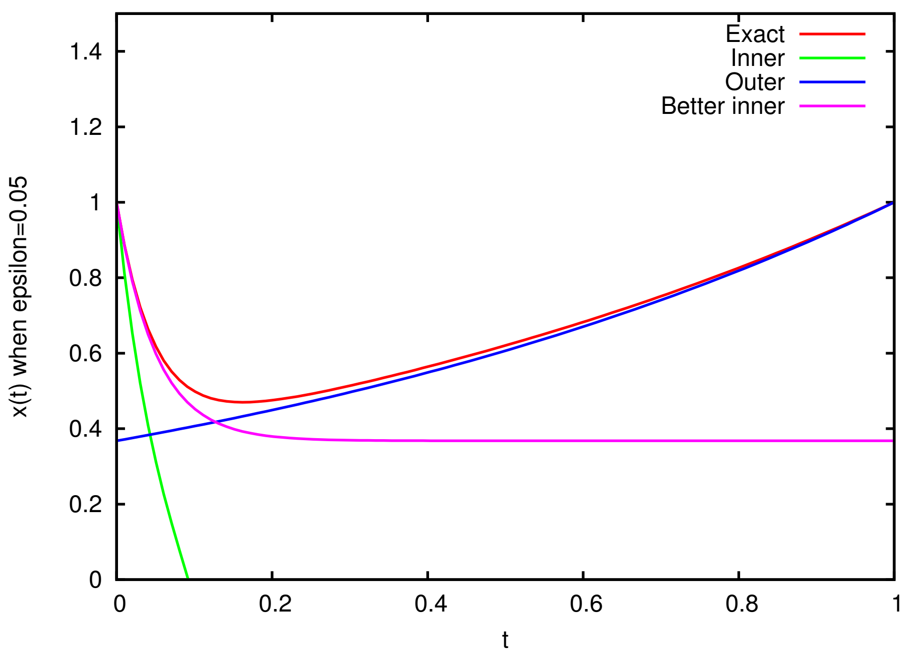
$x_{inner}(0) = C_1 + C_2 = 1$ implies $C_2 = 1 - C_1$. $x_{outer}(1) = 1$ implies $x_{outer}(t) = e^{t-1}$. $x_{inner}(\epsilon) = x_{outer}(\epsilon)$ implies $C_1 + (1 - C_1)e^{-1} = e^{\epsilon - 1}$. Solving, we find $C_1 = (e^{\epsilon-1} - e^{-1})/(1 - e^{-1})$.

so

$$x(t) \approx \begin{cases} \frac{\mathrm{e}^{\epsilon}-1}{e-1}(1 - e^{t/\epsilon}) + e^{-t/\epsilon} & \text{if } 0 \le t < \epsilon \\ e^{t-1} & \text{if } \epsilon \le t \le 1 \end{cases}$$

(f) Plot the exact and approximate solutions for $\epsilon = 0.05$.

Answer:



The matching condition given is a good naive starting point, but does not converge particularly well, as the diagram shows. In fact, it probably isn't a uniform approximation as $\epsilon \to \infty$. A better condition is to take $x_{inner}(\infty) = x_{outer}(0)$. The theory of determining the best matching condition for boundary-layer expansions like this one can be quite complex.

2. The maximum height h of a projectile is modelled as a function of the projectiles mass m, the initial velocity v, and the gravitational acceleration g:

$$h = \beta(m, v, g)$$

. The dimensions of the system's state are length, time, and mass. Inserting conversion factors Π_L, Π_T, and Π_M for length, time, and mass respectively, we find

$$h\Pi_L = \beta(m\Pi_M, v\Pi_L\Pi_T^{-1}, g\Pi_L\Pi_T^{-2}).$$

   (a) Find choices of these conversion factors to show

$$h = \alpha\frac{v^2}{g}$$

   for some constant α.

   (b) Show that we do not obtain a simple polynomial formula for the maximum height h if our model

$$h = \zeta(m, v, g, d)$$

   also incorporates the density of the atmosphere, d, with dimensions of mass per length cubed.

3. Another way to think of dimensional analysis is that since since scientific results should not depend on the units chosen for the variables, every good continuum-mechanics model should be expressible in terms of dimensionless variables (variables where all the units have cancelled out).

   A fluid-dynamics model might relate the drag force f on a projectile to radius of the projective r, the velocity v, the density of air d, and the dynamic viscosity m. Force has dimensions of mass by length per time squared, while dynamic viscosity has dimensions mass per length per time.

   (a) Find 3 linear equations that must be satisfied by the vector $\vec{n} = [n_f, n_r, n_v, n_d, n_m]$ to make

$$f^{n_f} r^{n_r} v^{n_v} d^{n_d} m^{n_m}$$

   dimensionless.

(b) Find a basis of the solution space of these linear equations. Use your basis to construct a set of dimensionless variables. (Note: This basis is not Unique! Different variables are used in different contexts.)

(c) Use linear algebra to show that every dimensionless polynomial in the original variables f, r, v, d, m can be expressed in terms of your dimensionless variables. In particular, express the Reynolds number $Re = dvr/m$ in terms of your dimensionless variables.

4. Supergiant stars have been observed to pulsate. The period of pulsation p has been modelled as a function of the star radius r, the star mass m, and the "universal" gravitational constant G.

   (a) Determine the dimensions of G.

   (b) Use dimensional analysis to construct a function model for p in terms of the other variables.

5. Use dimensional analysis to make dimensionless versions of the following differential equations:

   (a) $\dot{N} = r(1 - N/K)$.

   (b) $\dot{S} = -\beta SI,\ \dot{I} = \beta SI - \gamma I$.