Math 511, Autumn 2010

Problem Set 9: Center Manifolds, Non-Hyperbolic Dynamics, and Simple Singular-Perturbation Problems

1. Problem 9 from Meiss, Section 5.7, slightly simplified:

\[ \frac{dx}{dt} = x^3 - 2xy, \]
\[ \frac{dy}{dt} = x^2 - y. \]

(a) Find the first few terms of power-series approximation of the center manifold through the origin.

**Answer:**
We first check and see that (0, 0) is a stationary solution of the system. The local Jacobian is

\[ \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}. \]

There is one stable eigenvalue, and a nullspace in the direction of \( \langle 1, 0 \rangle \). To do our center-manifold approximation, we assume \( y \) is a function of \( x \) \( (y = g(x)) \) locally, so

\[ \dot{y} = g'(x) \dot{x}. \]

Taking \( g(x) = x^n(\alpha + \beta x + O(x^2)) \),

\[ x^2 - x^n(\alpha + \beta x) \approx x^{n-1}(n\alpha + (n + 1)\beta x)(x^3 - 2x^{n+1}(\alpha + \beta x)) \]

The monomials of this equation have degrees 2, \( n, n+1, n+2, n+2, 2n, 2n+1, \) and \( 2n+2 \). The smallest consistent choice for \( n \) is \( n = 2 \), as all other monomials then have greater or equal degree, and solving for \( \alpha \), we find \( \alpha = 1 \). The next highest degree is 3, but there is only one monomial of this degree, implying \( \beta = 0 \). If we extend our series to \( g(x) = x^2(\alpha + \beta x + \gamma x^2 + \delta x^4 + O(x^5)) \), we find the next two coefficients \( \gamma = 2 \) and \( \delta = 16 \). Thus, our center manifold is approximately \( y = g(x) = x^2(1 + 2x^2 + 16x^4) \).

(b) Use the Jacobian and the center-manifold to classify the stability of the origin.
Answer:
Along the center-manifold,
\[ \dot{x} = x^3 - 2xy = -x^3 - 4x^5 - 32x^7 + O(x^8). \]

Then \( x \) approaches 0 along the center manifold. Second, we observed above that the there was one negative eigenvalue. Thus, \((0, 0)\) is a locally-stable stationary solution.

2. Differential equations can also be defined for complex variables. If \( z(t) \in \mathbb{C} \), the ordinary differential equation
\[ \frac{dz}{dt} = z^4 \]
corresponds to a system of two real-valued nonlinear equations
\[ \frac{dx}{dt} = x^4 + y^4 - 6x^2y^2 \]
\[ \frac{dy}{dt} = 4x^3y - 4xy^3 \]
where \( z = x + iy \) (You should check this for yourself).

(a) Show that \( z = 0 \) is a non-hyperbolic stationary solution.

Answer:
The Jacobian is
\[ J = \begin{bmatrix} 4x^3 - 12xy^2 & 4y^3 - 12x^2y \\ 12x^2y - 4y^3 & 4x^3 - 12xy^2 \end{bmatrix}. \]

This vanishes completely at \( x + iy = 0 \), so all eigenvalues are 0. A stationary point is hyperbolic only if none of the eigenvalues have vanishing real part, so this point is not hyperbolic.

(b) Transform the equations to polar coordinates using \( z(t) = r(t)e^{\theta i} \) to find 3 center-manifold solutions through \( z = 0 \).

Answer:
Plugging this ansatz into our original equation, we find we must have
\[ e^{\theta i} \dot{r} = e^{4\theta i} r. \]

This can only be satisfied if
\[ e^{4\theta i - \theta i} \]
is a real number, implying $3\theta$ is an integer multiple of $\pi$. The 3 distinct cases are then $\theta \in \{0, \pi/3, 2\pi/3\}$. So then there are 3 lines through 0 which are solutions. Since 0 is not hyperbolic, these all must be center manifolds.

(c) Show that $z = 0$ is a (non-hyperbolic) saddle-point.

**Answer:**

If we move the angle and study the reduced equation

\[ \dot{r} = e^{3\theta} r^4 \]

now, we see that the change in radius is monotone along each line. If $\theta = 0$, $r$ increases. If $\theta = \pi/3$, $r$ is decreasing, and if $\theta = 2\pi/3$, $r$ is increasing. So we have a saddle point.