Math 511, Autumn 2010

Problem Set 9: Center Manifolds, Non-Hyperbolic Dynamics, and Simple Singular-Perturbation Problems

1. Problem 9 from Meiss, Section 5.7, slightly simplified:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^3 - 2xy,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x^2 - y.$$

   (a) Find the first few terms of power-series approximation of the center manifold through the origin.

   Answer:
   We first check and see that $(0,0)$ is a stationary solution of the system. The local Jacobian is

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

   There is one stable eigenvalue, and a nullspace in the direction of $\langle 1, 0 \rangle$. To do our center-manifold approximation, we assume y is a function of x ($y = g(x)$) locally, so

$$\dot{y} = g'(x)\dot{x}.$$

   Taking $g(x) = x^n(\alpha + \beta x + O(x^2))$,

$$x^2 - x^n(\alpha + \beta x) \approx x^{n-1}(n\alpha + (n+1)\beta x)(x^3 - 2x^{n+1}(\alpha + \beta x))$$

   The monomials of this equation have degrees 2, n, $n+1$, $n+2$, $n+2$, $2n$,$2n+1$, and $2n+2$. The smallest consistent choice for n is $n = 2$, as all other monomials then have greater or equal degree, and solving for α, we find $\alpha = 1$. The next highest degree is 3, but there is only one monomial of this degree, implying $\beta = 0$. If we extend our series to $g(x) = x^2(\alpha + \beta x + \gamma x^2 + \delta x^4 + O(x^5))$, we find the next two coefficients $\gamma = 2$ and $\delta = 16$. Thus, our center manifold is approximately $y = g(x) = x^2(1 + 2x^2 + 16x^4)$.

   (b) Use the Jacobian and the center-manifold to classify the stability of the origin.

Answer:
Along the center-manifold,

$$\dot{x} = x^3 - 2xy = -x^3 - 4x^5 - 32x^7 + O(x^8).$$

Then x approaches 0 along the center manifold. Second, we observed above that the there was one negative eigenvalue. Thus, $(0, 0)$ is a locally-stable stationary solution.

2. Differential equations can also be defined for complex variables. If $z(t) \in \mathbb{C}$, the ordinary differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = z^4$$

corresponds to a system of two real-valued nonlinear equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^4 + y^4 - 6x^2y^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4x^3y - 4xy^3$$

where $z = x + iy$ (You should check this for yourself).

(a) Show that $z = 0$ is a non-hyperbolic stationary solution.

Answer:
The Jacobian is

$$J = \begin{bmatrix} 4x^3 - 12xy^2 & 4y^3 - 12x^2y \\ 12x^2y - 4y^3 & 4x^3 - 12xy^2 \end{bmatrix}.$$

This vanishes completely at $x+iy = 0$, so all eigenvalues are 0. A stationary point is hyperbolic only if none of the eigenvalues have vanishing real part, so this point is not hyperbolic.

(b) Transform the equations to polar coordinates using $z(t) = r(t)e^{\theta i}$ to find 3 center-manifold solutions through $z = 0$.

Answer:
Plugging this ansatz into our original equation, we find we must have

$$e^{\theta i}\dot{r} = e^{4\theta i}r.$$

This can only be satisfied if

$$e^{4\theta i - \theta i}$$

is a real number, implying 3θ is an integer multiple of π. The 3 distinct cases are then $\theta \in \{0, \pi/3, 2\pi/3\}$. So then there are 3 lines through 0 which are solutions. Since 0 is not hyperbolic, these all must be center manifolds.

(c) Show that $z = 0$ is a (non-hyperbolic) saddle-point.

Answer:

If we move the angle and study the reduced equation

$$\dot{r} = e^{3\theta i} r^4$$

now, we see that the change in radius is monotone along each line. If $\theta = 0$, r increases. If $\theta = \pi/3$, r is decreasing, and if $\theta = 2\pi/3$, r is increasing. So we have a saddle point.