Math 511, Autumn 2010

Problem Set 8, Stroboscopic maps and local stability criteria

1. One of the most widely studied ODE's with periodic coefficients is Hill's equation

$$\ddot{x} + f(t)x = 0, \text{with} f(0) = f(nT) \forall n \in \mathbb{Z}.$$

It is used to study phenomena like http://www.youtube.com/watch?v=cHTibqThCTU The Meissner equation is the simplest case. Take $T = t_1 + t_2$, $f(t) = a_1$ if $0 \le t < t_1$, $f(t) = a_2$ if $t_1 \le t < T$.

(a) Write Hill's equation as a first-order system

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -f(t) & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(b) Calculate the matrix exponentials $\exp(B(0)t_1)$ and $\exp(B(t_1)t_2)$. Assume $a_i > 0$ for each i.

Answer:
Using the Cayley--Hamilton theorem method or our previous problems, we can rewrite the matrix exponentials

$$e^{\begin{bmatrix} 0 & 1 \\ -a_1 & 0 \end{bmatrix} t} = \begin{bmatrix} \cos(\sqrt{a_1}t) & \frac{1}{\sqrt{a_1}}\sin(\sqrt{a_1}t) \\ -\sqrt{a_1}\sin(\sqrt{a_1}t) & \cos(\sqrt{a_1}t) \end{bmatrix}$$

$$e^{\begin{bmatrix} 0 & 1 \\ -a_2 & 0 \end{bmatrix} t} = \begin{bmatrix} \cos(\sqrt{a_2}t) & \frac{1}{\sqrt{a_2}}\sin(\sqrt{a_2}t) \\ -\sqrt{a_2}\sin(\sqrt{a_2}t) & \cos(\sqrt{a_2}t) \end{bmatrix}$$

(c) Calculate the monodromy matrix $\Psi = \exp(B(t_1)t_2)\exp(B(0)t_1)$.

Answer:

$$\Psi = \begin{bmatrix} \cos(\sqrt{a_2}t_2) & \frac{1}{\sqrt{a_2}}\sin(\sqrt{a_2}t_2) \\ -\sqrt{a_2}\sin(\sqrt{a_2}t_2) & \cos(\sqrt{a_2}t_2) \end{bmatrix} \begin{bmatrix} \cos(\sqrt{a_1}t_1) & \frac{1}{\sqrt{a_1}}\sin(\sqrt{a_1}t_1) \\ -\sqrt{a_1}\sin(\sqrt{a_1}t_1) & \cos(\sqrt{a_1}t_1) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\sqrt{a_2}t_2)\cos(\sqrt{a_1}t_1) - \frac{\sqrt{a_1}}{\sqrt{a_2}}\sin(\sqrt{a_2}t_2)\sin(\sqrt{a_1}t_1) & \frac{1}{\sqrt{a_1}}\cos(\sqrt{a_2}t_2)\sin(\sqrt{a_1}t_1) + \frac{1}{\sqrt{a_2}}\cos(\sqrt{a_1}t_1)\sin(\sqrt{a_2}t_2) \\ -\sqrt{a_2}\cos(\sqrt{a_1}t_1)\sin(\sqrt{a_2}t_2) - \sqrt{a_1}\cos(\sqrt{a_2}t_2)\sin(\sqrt{a_1}t_1) & \cos(\sqrt{a_2}t_2)\cos(\sqrt{a_1}t_1) - \frac{\sqrt{a_2}}{\sqrt{a_1}}\sin(\sqrt{a_2}t_2)\sin(\sqrt{a_1}t_1) \end{bmatrix}$$

(d) Show that $\det(\Psi) = 1$.

Answer:
Since $\det(AB) = \det(A)\det(B)$, and $\cos^2 x + \sin^2 x = 1$, we know from the individual matrices that $\det(\Psi) = 1$ also.

(e) The eigenvalues of the monodromy matrix are called Floquet multipliers. Find a condition for all the Floquet multipliers of the monodromy matrix to be within the unit circle.

Answer:
The criteria for Lagrange stability based on a monodromy matrix with determinate 1 is $-2 < \operatorname{tr}(\Psi) < 2$. Here, this means the same thing as $\left[\cos(\sqrt{a_2}t_2)\cos(\sqrt{a_1}t_1) - \frac{\sqrt{a_2}}{\sqrt{a_1}}\sin(\sqrt{a_2}t_2)\sin(\sqrt{a_1}t_1)\right]^2 < 1$.

2. Find one more condition on the parameters of

$$\begin{bmatrix} -2\alpha & 0 & -2 \\ \alpha+\beta & -\beta & 1 \\ 0 & -\gamma\alpha & -\alpha-\beta \end{bmatrix}$$

so that when $0 < \alpha$, $0 < \beta$, $0 < \gamma$, then all the eigenvalues of the matrix lie in the left half-plane .

Answer:
The characteristic polynomial comes out as

$$z^3 + (2\,\beta + 3\,\alpha)\,z^2 + \left(2\,\alpha^2 + \beta^2 + \gamma\,\alpha + 5\,\beta\,\alpha\right)z - 2\,\beta\,\alpha\,(-\beta + \gamma - \alpha) = 0$$

The Routh--Hurwitz criteria for all the eigenvalues to have negative real part is $(2\,\beta + 3\,\alpha) > 0$, $-2\,\beta\,\alpha\,(-\beta+\gamma-\alpha) > 0$ $17\,\beta\,\alpha^2 + 2\,\beta^3 + 4\,\gamma\,\alpha\,\beta + 11\,\alpha\,\beta^2 + 6\,\alpha^3 + 3\,\gamma\,\alpha^2 > 0$ The first and third conditions are always satisfied, but the second is only satisfied if $\gamma < \alpha + \beta$.