# Math 511, Autumn 2010

## Problem Set 6, 2D Nonlinear systems -- $\omega$-limit sets, Limit cycles, and Bifurcations

This homework is a bit longer than normal. Feel free to ask for hints if you get stuck.

1. Consider the following system of ODE's that describes a 1-resource competition model:

$$\dot{r} = n - axr - byr,$$
$$\dot{x} = axr - dx$$
$$\dot{y} = byr - fy.$$

   We will make the common assumption that all parameters have positive values.

   (a) Use the assumption that $r = n/(ax+by)$ to reduce this system from 3 equations to two equations. (This is called a quasi-steady-state approximation. We will talk about it more in the future.)
   **Answer:**
   The system is transformed to

$$\dot{x} = \frac{axn}{ax+by} - dx, \dot{y} = \frac{byn}{ax+by} - fy.$$

   (b) Show that the system from part (a) can not have limit-cycle solutions in the positive quadrant ($x \geq 0$, $y \geq 0$) by finding a Dulac function $B(x, y)$ that allows us to apply the Bendixson--Dulac negative criteria.
   **Answer:**
   In previous examples, the Dulac function $B(x, y) = 1/xy$ worked. If we try it here, it doesn't quite work because of the $ax + by$ in the denominators of the ODE system. If, however, we try $B(x, y) = (ax + by)/xy$, then the divergence

$$\frac{\partial(B\dot{x})}{\partial x} + \frac{\partial(B\dot{y})}{\partial y} = -\frac{da}{y} - \frac{fb}{x} < 0$$

   The Bendixson--Dulac negative criteria now implies that there can not be any limit cycles.

   (c) Determine the $\omega$-limits for all initial conditions with atleast one positive coordinate, assuming (without loss of generality) that $a/d > b/f$.
   **Answer:**
   Under the specified parameter condition the stationary solution $(n/d, 0)$ will be the omega limit of all positive initial conditions and all initial conditions where $x > 0, y = 0$. For initial conditions with $y > 0, x = 0$, $(0, n/f)$ is the omega-limit.

2. One way to prove a stable limit-cycle exists within a simply-connected (no holes) closed and bounded domain is to show that there (1) the vectors point inward at every point on the boundary, and (2) that all the stationary solutions inside the domain are unstable foci or nodes. Then the Poincare--Bendixson theorem implies that each nonstationary initial conditions in the domain must have a limit cycle as its $\omega$-limit set.

   Show that

$$\dot{x}_1 = a - x_1 + x_1^2 x_2$$
$$\dot{x}_2 = b - x_1^2 x_2$$

   has a limit cycle inside the quadrilateral

$$[(a+b+b/a^2+\delta, 0), (a, 0), (a, b/a^2), (a+b+\delta, b/a^2)],$$

   if $\delta$ is a small positive number, $a = 0.1$, and $b = 0.3$.

   **Answer:**
   First, we show the region is invariant. For any point on the line between $(a+b+b/a^2+\delta, 0)$ and $(a, 0)$, $\dot{x}_2 = b > 0$. For any point on the line between $(a, 0)$ and $(a, b/a^2)$, $\dot{x}_1 = a^2 x_2 > 0$. For any point on the segment between $(a, b/a^2)$ and $(a+b+\delta, b/a^2)$, $\dot{x}_2 \leq b - (a^2)(b/a^2) = 0$. In the special case where $\dot{x}_2 = 0$, inspection shows this solution enters the invariant region. Finally, if $t \in [0, 1]$, the line-segment $(a+b+(1-t)b/a^2+\delta, tb/a^2)$ has slope $-1$, and $\dot{x}_1 + \dot{x}_2 = a + b - (a+b+(1-t)b/a^2+\delta) = -(1-t)b/a^2 - \delta < 0$. So nothing flows out of this region.

   There is one stationary solution $(x_1, x_2) = (a, b/a^2)$. The Jacobian of the system is

$$J(x_1, x_2) = \begin{bmatrix} -1 + 2x_1 x_2 & x_1^2 \\ -2x_1 x_2 & -x_1^2 \end{bmatrix}$$
$$J(a, b/a^2) = \begin{bmatrix} -1 + 2b/a & a^2 \\ -2b/a & -a^2 \end{bmatrix}$$
$$J = \begin{bmatrix} 5 & 0.01 \\ -6 & -0.01 \end{bmatrix}, \quad \mathrm{tr}\,(J) > 0, \det\,(J) = 0.01 > 0$$

   so by the Routh--Hurwitz criteria, the stationary solution must be locally unstable. There must be omega-limits for initial conditions in this region other than the stationary solution, so by the Poincare--Bendixson theorem, there must be a limit cycle.