Math 511, Autumn 2010

Problem Set 5, 2D Nonlinear systems -- Constants of Motion

1. Consider the equation $\ddot{\theta} = -\theta + \epsilon\theta^3$, where $\epsilon > 0$. This equation is a ‘‘weakly’’ nonlinear version of the harmonic oscillator.

   (a) Transform this 2nd-order equation to a system of two 1st-order equations.
   Answer:

   $$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -u + \epsilon u^3 \end{bmatrix}$$

   (b) Find all stationary solutions.
   Answer:
   The stationary solutions are $(u, v) \in \{(0, 0), (\pm 1/\sqrt{\epsilon}, 0)\}$.

   (c) Determine the local stability of the stationary solutions, if possible.
   Answer:
   The jacobian is $\begin{bmatrix} 0 & 1 \\ 3\epsilon u^2 - 1 & 0 \end{bmatrix}$. At $(0, 0)$, the eigenvalues are $\pm i$, so that stationary solution is not hyperbolic, and we can not determine the local stability from the linearization. At the two other equilibria, the Jacobian comes at as

   $$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

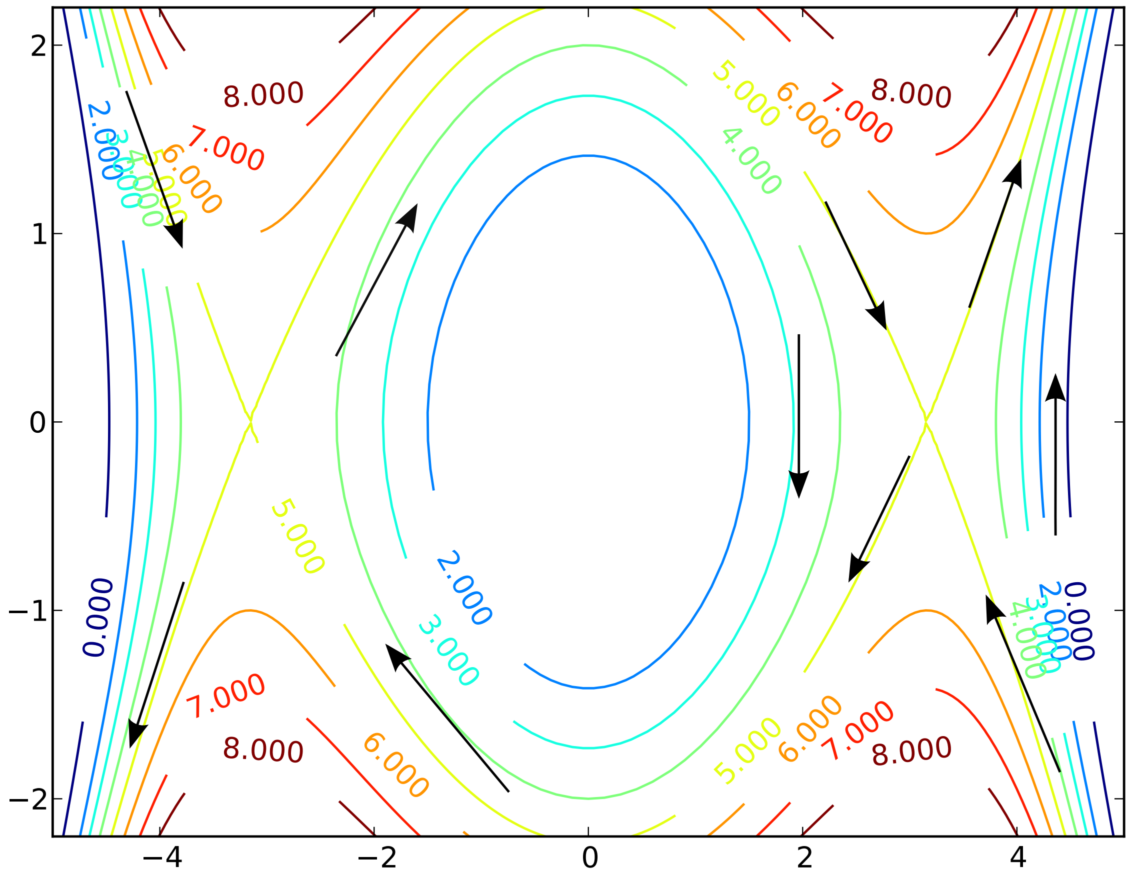
   so the eigenvalues are $\pm\sqrt{2}$. These points are hypberbolic saddle points.

   (d) Find a constant-of-motion for this system.
   Answer:
   From our system $\dot{v}/\dot{u} = dv/du = (-u + \epsilon u^3)/v$. Rearranging, we get the exact differential $vdv + (u - \epsilon u^3)du = 0$ which is integrated to yield the constant-of-motion

   $$C = \frac{1}{2}v^2 + \frac{1}{2}u^2 - \frac{\epsilon}{4}u^4.$$

(e) Use the constant-of-motion and the stationary solutions to draw a phase-plane portrait of the system dynamics.

Answer:



(f) Classify the possible asymptotic behaviors of solutions as a function of the initial conditions.

Answer:
There are 3 distinct asymptotic behaviors. $u(t), v(t)$ may diverge to $(+\infty, +\infty)$, $(-\infty, -\infty)$, or stay bounded and oscillate periodically if the initial condition is in the invariant region around the origin. There are also stable manifolds leading into the saddle points, but these are unstable under small perturbations.