Math 511, Autumn 2010

Problem Set 4, Linear systems with Constant Matrices

1. Suppose we have an $m \times m$ matrix A, and a vector $w \in \mathbb{R}^m$ such that

$$(A - rI)^2 w = -\zeta^2 w$$

for some real numbers r and ζ. Show that

$$x(t) = \alpha e^{rt}[\cos(\zeta t)(A - rI)w - \zeta \sin(\zeta t)w] + \beta e^{rt}[\sin(\zeta t)(A - rI)w + \zeta \cos(\zeta t)w]$$

is a solution of $\dot{x} = Ax$. (Note: $r \pm \zeta i$ are complex eigenvalues, in common parlance)

Answer:
Substituting in our proposed solution, we should have

$$-\mathrm{e}^{rt} w \left(r^2 - 2\,Ar + A^2 + \zeta^2\right) \cos\left(\zeta\, t\right) \alpha - \mathrm{e}^{rt} w \left(r^2 - 2\,Ar + A^2 + \zeta^2\right) \beta \,\sin\left(\zeta\, t\right)$$

From the relation above, we know both coefficients vanish, so this must be a solution.

2. Find a general solution of $\dot{x} = Bx$ if

$$B = \begin{bmatrix} -1 & -36 & -12 \\ 0 & -1 & 0 \\ 0 & 12 & 3 \end{bmatrix}. \tag{1}$$

Answer:
The eigenvalues of B are -1 and 3, which an be found from the characteristic polynomial or inspection. Calculating eigenvectors, we find

$$x(t) = c_1 \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} e^{3t} \tag{2}$$

3. Find a general solution of $\dot{x} = Cx$ if

$$C = \begin{bmatrix} -1 & 1 \\ -4 & -1 \end{bmatrix}. \tag{3}$$

Answer:
Observe that $(C + I)^2 = -4I$, so we can apply the formula from problem 1 with $r = -1$ and $\zeta = 2$.

$$x(t) = \alpha e^{-t}[\cos(2t)(C + I) - 2\sin(2t)]w + \beta e^{-t}[\sin(2t)(C + I) + 2\cos(2t)]w$$

Without loss of generality, take $w = [1, 0]^T$.

$$x(t) = C_1 e^{-t} \begin{bmatrix} -2\sin(2t) \\ -4\cos(2t) \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 2\cos(2t) \\ -4\sin(2t) \end{bmatrix}. \tag{4}$$

Since there are two undetermined coefficients, and the original system was 2-dimensional, this solution has to span the solution space for various choices of C_1 and C_2.