Math 511, Autumn 2010

Problem Set 3, Series methods

1. Use the MacLaurin series method discussed in class to construct solutions of the homogeneous Airy equation

$$y'' - xy = 0.$$

   - You need only find the first 4 terms in your solutions.
     Answer:
     If $y = \sum_{n=0}^{\infty} a_n x^n$, then

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$2 \times 1 \times a_2 + \sum_{m=1}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty} a_{m-1}x^m = 0$$

     Thus, we find $a_2 = 0$, and $a_{m+2} = a_{m-1}/(m+2)(m+1)$. Taking $m-1 = k$, $a_{k+3} = a_k/(k+3)(k+2)$.

$$y(x) = a_0(1 + \frac{1}{6}x^3 + \frac{1}{6 \times 30}x^6 \ldots) + a_1(x + \frac{1}{12}x^4 + \frac{1}{12 \times 42}x^7 \ldots).$$

2. **(H)** Consider the following equation

$$y'' + \frac{y'}{x} - \left(1 + \frac{9}{x^2}\right) y = 0$$

   (a) What happens when you look for a MacLaurin series solutions $y(x) = \sum_{n=0}^{\infty} a_n x^n$?
   Answer:
   In trying to match powers, we find that all the coefficients in the series vanish. We obtain only one solution

$$y(x) = a_3(x^3 + \frac{1}{16}x^5 + \frac{1}{640}x^7 \ldots)$$

   (b) What happens when you look for Frobenius series solutions $y(x) = x^v \sum_{n=0}^{\infty} a_n x^n$?
   Answer:
   Using a Frobenius series, we find that the highest order term is $(v-3)(v+3)a_0 = 0$. This is satisfied by $v = 3$, corresponding to our first series solution, but also by $v = -3$. Taking $v = -3$, we find another solution

$$\frac{a_0}{x^3}(1 - x^2/8 + x^4/64 \ldots)$$