Math 511, Autumn 2010

Problem Set 2, Introduction to Linear Equations

1. Linear operators. Let D_x represent the derivative of a function with respect to x, and similarly for any subscript. and I represent the identity operator. The commutator $[A, B]$ of two linear operators A and B is $AB - BA$.

   (a) Find $[D_x, x]$

   Answer:

   It is convenient to introduce a dummy function $f(x)$ for bookkeeping.

$$
\begin{aligned}
[D_x, x]f(x) &= D_x(xf(x)) - xD_xf(x) \\
&= f(x) + xD_xf(x) - xD_xf(x) = f(x)
\end{aligned}
$$

   so $[D_x, x] = 1$, the identity operator.

   (b) Find $[D_x - x, D_x + x]$

   Answer:

$$
\begin{aligned}
[D_x - x, D_x + x]f(x) &= (D_x - x)(D_x + x)f - (D_x + x)(D_x - x)f \\
&= (D_x - x)(f' + xf) - (D_x + x)(f' - xf) \\
&= D_x(f' + xf) - x(f' + xf) - D_x(f' - xf) - x(f' - xf) \\
&= (f'' + f + xf') - (xf' + x^2 f) - (f'' - f - xf') - (xf' - x^2 f) \\
&= 2f
\end{aligned}
$$

   So $[D_x - x, D_x + x] = 2$

   (c) Find $[f(x) + D_x^2, g(x) + D_x x]$.

   Answer:

   After lengthy and careful calculation, $g''(x) + 2g'(x)D_x + 2D_x^2 - xf'(x)$.

2. Find the nullspace of the following linear homogeneous equation.

$$
y''' - 3y'' - 45y' + 175y = 0
$$

   Answer:

   The characteristic equation $z^3 - 3z^2 - 45z + 175 = 0$ factors as $(z+7)(z-5)^2 = 0$. You can find this, for instance, by plotting. 5 is a repeated eigenvalue. The nullspace consists of functions

$$
y(x) = c_1 e^{-7x} + c_2 e^{5x} + c_3 x e^{5x}
$$

   for arbitrary constants c_1, c_2, c_3.

3. Find a linear operator with nullspace

$$c_0 e^t + c_1 e^{-t}\sin(4t) + c_2 e^{-t}\cos(4t)$$

Answer:
The characteristic polynomial $(z-1)(z+1-4i)(z+1+4i) = z^3+z^2+15z-17$ so

$$y''' + y'' + 15y' - 17y = 0$$

has this as a nullspace.

4. Find an equation for the maximum amplitude of oscillations of the damped harmonic oscillator with sinusoidal forcing

$$\ddot{x} = -p\dot{x} - qx + \sin(\omega t), p > 0, q > 0.$$

Answer:
Using the ansatz $x(t) = A\sin\omega t + B\cos\omega t$, we substitute, solve for the coefficients, and we find this equation has a particular solution

$$y_p(t) = \frac{\sin(\omega t)(-\omega^2+q) - p\omega\cos(\omega t)}{p^2\omega^2 + \omega^4 - 2\,\omega^2 q + q^2}$$

Differentiating, we find the extreme values occur at $t = -\frac{1}{\omega}\arctan\left(\frac{-\omega^2+q}{p\omega}\right)$ in which case the amplitude is given by

$$\frac{1}{\sqrt{p^2\omega^2 + \omega^4 - 2\,\omega^2 q + q^2}}$$

5. Boundary-value problems are ordinary differential equations where conditions on the solution are given at both the start and the stop times. Two-sided Green functions are used to solve inhomogeneous linear boundary-value problems. Consider

$$y'' = (x+1)(x-1),\ y(-1) = 2,\ y(1) = 1$$

(a) Find a initial-value Green function by solving

$$y_i'' = 0, y_i(-1) = a, y_i(1) = 0.$$

Answer:
$y_i(x) = a - a(x+1)/2$

(b) Find a terminal-value Green function by solving

$$y_f'' = 0, y_f(-1) = 0, y_f(1) = b.$$

Answer:
$y_f(x) = b + b(x-1)/2$

(c) Find the two-sided Green function satisfying

$$y''_G = \delta(x-s), y_G(-1) = 0, y_G(1) = 0.$$

You may find it convenient to use absolute-value when writing the function.
Answer:
Integrating around $x = s$,

$$\int_{s-\epsilon}^{s+\epsilon} y''_G(x)dx = 1 = y'_G(s+\epsilon) - y'_G(s-\epsilon).$$

Applying all conditions and solving for unknowns,

$$y_G(x;s) = \begin{cases} \frac{1}{2}(x+1)(s-1) & \text{if } x < s, \\ \frac{1}{2}(x-1)(s+1) & \text{if } x \geq s, \end{cases}$$

(d) Use these to solve the full inhomogeneous equation with boundary conditions given above.
Answer:

$$y(x) = 2y_i(x) + 1y_f(x) + \int_{-1}^{1} y_G(x;s)(s^2-1)ds \quad (1)$$

$$= 2 - (x+1) + 1 + (x-1)/2 + \int_{-1}^{1} y_G(x;s)(s^2-1)ds \quad (2)$$

$$= 3/2 - x/2 + \int_{x}^{1} \frac{1}{2}(x+1)(s-1)(s^2-1)ds + \int_{-1}^{x} \frac{1}{2}(x+1)(s-1)(s^2-1)ds \quad (3)$$

$$= 1/12\ (x-1)(x+1)\left(x^2-5\right) + 3/2 - 1/2\,x \quad (4)$$