Math 511, Autumn 2010

Problem Set 2, Introduction to Linear Equations

1. Linear operators. Let $D_x$ represent the derivative of a function with respect to $x$, and similarly for any subscript. and $I$ represent the identity operator. The commutator $[A,B]$ of two linear operators $A$ and $B$ is $AB - BA$.

(a) Find $[D_x,x]$  
**Answer:** 
It is convenient to introduce a dummy function $f(x)$ for bookkeeping.

$$ [D_x,x]f(x) = D_x(xf(x)) - xD_xf(x) $$

$$ = f(x) + xD_xf(x) - xD_xf(x) = f(x) $$

so $[D_x,x] = 1$, the identity operator.

(b) Find $[D_x - x, D_x + x]$ 
**Answer:**

$$ [D_x - x, D_x + x]f(x) = (D_x - x)(D_x + x)f - (D_x + x)(D_x - x)f $$

$$ = (D_x - x)(f' + xf) - (D_x + x)(f' - xf) $$

$$ = D_x(f' + xf) - x(f' + xf) - D_x(f' - xf) - x(f' - xf) $$

$$ = (f'' + f + xf') - (xf' + x^2f) - (f'' - f - xf') - (xf' - x^2f) $$

$$ = 2f $$

So $[D_x - x, D_x + x] = 2$

(c) Find $[f(x) + D_x^2, g(x) + D_x x]$.

**Answer:** 
After lengthy and careful calculation, $g''(x) + 2g'(x)D_x + 2D_x^2 - xf'(x)$.

2. Find the nullspace of the following linear homogeneous equation.

$$ y''' - 3y'' - 45y' + 175y = 0 $$

**Answer:** 
The characteristic equation $z^3 - 3z^2 - 45z + 175 = 0$ factors as $(z + 7)(z - 5)^2 = 0$. You can find this, for instance, by plotting. 5 is a repeated eigenvalue. The nullspace consists of functions

$$ y(x) = c_1 e^{-7x} + c_2 e^{5x} + c_3 xe^{5x} $$

for arbitrary constants $c_1,c_2,c_3$. 

3. Find a linear operator with nullspace
\[ c_0e^t + c_1e^{-t}\sin(4t) + c_2e^{-t}\cos(4t) \]

**Answer:**
The characteristic polynomial \((z - 1)(z + 1 - 4i)(z + 1 + 4i) = z^3 + z^2 + 15z - 17\) so
\[ y''' + y'' + 15y' - 17y = 0 \]
has this as a nullspace.

4. Find an equation for the maximum amplitude of oscillations of the damped harmonic oscillator with sinusoidal forcing
\[ \ddot{x} = -p\dot{x} - qx + \sin(\omega t), p > 0, q > 0. \]

**Answer:**
Using the ansatz \(x(t) = A\sin\omega t + B\cos\omega t\), we substitute, solve for the coefficients, and we find this equation has a particular solution
\[ y_p(t) = \frac{\sin(\omega t)}{p^2\omega^2 + \omega^4 - 2\omega^2q + q^2} \]
Differentiating, we find the extreme values occur at \(t = -\frac{1}{\omega}\arctan\left(\frac{-\omega^2 + q}{p\omega}\right)\) in which case the amplitude is given by
\[ \frac{1}{\sqrt{p^2\omega^2 + \omega^4 - 2\omega^2q + q^2}} \]

5. Boundary-value problems are ordinary differential equations where conditions on the solution are given at both the start and the stop times. Two-sided Green functions are used to solve inhomogeneous linear boundary-value problems. Consider
\[ y'' = (x + 1)(x - 1), \ y(-1) = 2, \ y(1) = 1 \]

(a) Find a initial-value Green function by solving
\[ y_i'' = 0, \ y_i(-1) = a, \ y_i(1) = 0. \]

**Answer:**
\[ y_i(x) = a - a(x + 1)/2 \]

(b) Find a terminal-value Green function by solving
\[ y_f'' = 0, \ y_f(-1) = 0, \ y_f(1) = b. \]

**Answer:**
\[ y_f(x) = b + b(x - 1)/2 \]
(c) Find the two-sided Green function satisfying
\[ y''_G = \delta(x - s), \quad y_G(-1) = 0, \quad y_G(1) = 0. \]
You may find it convenient to use absolute-value when writing the function.

**Answer:**
Integrating around \( x = s \),
\[ \int_{s-\epsilon}^{s+\epsilon} y''_G(x) \, dx = 1 = y'_G(s + \epsilon) - y'_G(s - \epsilon). \]

Applying all conditions and solving for unknowns,
\[ y_G(x; s) = \begin{cases} 
\frac{1}{2}(x + 1)(s - 1) & \text{if } x < s, \\
\frac{1}{2}(x - 1)(s + 1) & \text{if } x \geq s,
\end{cases} \]

(d) Use these to solve the full inhomogeneous equation with boundary conditions given above.

**Answer:**
\[ y(x) = 2y_i(x) + 1y_f(x) + \int_{-1}^{1} y_G(x; s)(s^2 - 1) \, ds \tag{1} \]
\[ = 2 - (x + 1) + 1 + (x - 1)/2 + \int_{-1}^{1} y_G(x; s)(s^2 - 1) \, ds \tag{2} \]
\[ = 3/2 - x/2 + \int_{x}^{1} \frac{1}{2}(x + 1)(s - 1)(s^2 - 1) \, ds + \int_{-1}^{x} \frac{1}{2}(x + 1)(s - 1)(s^2 - 1) \, ds \tag{3} \]
\[ = 1/12 \ (x - 1) \ (x + 1) \ (x^2 - 5) + 3/2 - 1/2 \ x \tag{4} \]