Math 511, Autumn 2010

Problem Set 1, First order equations

1. Suppose we have a set of lines $Y = \{y(x;a) = (x-a)^2 - a : a \in \mathbb{R}\}$

   (a) Find a differential equation that is independent of a and for which each element of Y is a solution.
   Answer:
   Differentiating with respect to x, we find $a = x - y'/2$, so

   $$\frac{1}{4}\left(\frac{dy}{dx}\right)^2 + \frac{1}{2}\frac{dy}{dx} - x - y = 0$$

   (b) Find a singular solution of this ODE by calculating the envelop of Y.
   Answer:
   $y = -x - 1/4$

2. Gilpin and Ayala (Proceedings of the National Academy of Sciences, 1973) studied a two-species generalization of the equation

   $$\frac{dN}{dt} = rN\left(\frac{k^\theta - N^\theta}{\theta k^\theta}\right).$$

   (a) Use a method from class to solve this ODE. **Answer:**
   This is a Bernoulli equation, so we can solve it using a substitution.

   (b) (**H**) Discussion the behavior of the ODE and it's solutions in the limit of $\theta \to 0$. In this limit, the model is called the Gompertz growth model.
   Answer:
   Observe that $(N/k)^\theta = \exp(\theta \ln(N/k))$ and apply l'Hopital's rule to fine the limit.

   $$\lim_{\theta \to 0^+}\left(\frac{k^\theta - N^\theta}{\theta k^\theta}\right)$$

3. (**H**) Determine when is $\mu(x,y) = x^p y^q$ an integrating factor for

   $$ax + by + Ax^m y^{n+1} + Bx^{m+1}y^n y' = 0.$$

   Answer:
   If μ is an integrating factor, it should turn this ODE into an exact equation.

   $$ax^{p+1}y^q + bx^p y^{q+1} + Ax^{p+m}y^{q+n+1} + Bx^{p+m+1}y^{q+n}y' = 0.$$

   The conditions yield $A(q+n+1) - B(p+m+1) = 0$, $aq = 0$, $b(q+1) = 0$ most of the time. If $m = 1, n = -1$, then $A(q+n+1) - B(p+m+1) + qa = 0$, $b(q+1) = 0$. If $m = n = 0$, the problem is linear and can be handled directly.

4. Find general solutions for the following first-order ordinary differential equations

(a)

$$\dot{x} = \frac{x^2 + 2t^2}{t\,(t + x)}$$

Answer:
The right-hand side is homogeneous function, so we use the change-of-variables $ut = x$ to make the equation separable. We can integrate to obtain a solution in implicit form

$$\frac{e^{-x/t}}{t(x/t - 2)^3} = K$$

for some unknown constant K.

(b)

$$\frac{dy}{dx} = y \sin(x) + y^2$$

Answer:
This is a Bernoulli equation, so we use the substitution $y = 1/v$ to obtain the linear equation $v' = -\sin tv - 1$. In general, a Bernoulli equation

$$\dot{y} = p(t)y + q(t)y^n$$

with $n \neq 1$ has solution

$$y(t) = \left\{ \frac{(1-n)\int_t e^{-(1-n)\int_v p(u)du} q(v)dv + C}{e^{-(1-n)\int_t p(u)du}} \right\}^{1/(1-n)}$$

$$y(t) = \frac{e^{-\cos t}}{(1-n)\int_t e^{-\cos v} dv + C}$$

(c)

$$y' = -x + x^2 - 2xy^2 + y^4$$

Answer:
I do not know of any closed-form solution for this equation. This is included as an illustration that even relatively-simple differential equations can fail to have elementary solutions.

(d)

$$\frac{dx}{dt} = -\frac{\tan t}{\tan x}$$

Answer:

$$\int \frac{\sin x}{\cos x} dx = \int \frac{\sin t}{\cos t} dt$$
$$-\ln \cos x = -\ln \cos t + C$$
$$\cos x = K \cos t$$

(e)

$$\frac{dy}{dx} = xy^3 + 2y^3 - 2x - 4$$

Answer:

$$y' = (x+2)(y^3 - 2)$$

$$\int \frac{dy}{y^3 - 2} = \frac{1}{2}\frac{1}{2}x^2 + 2x + C$$