Homework 7, Math 451, Spring 2013

due Monday, March 25th

This home covers material from chapter 7 of Mathews and Fink.

1. Consider the integral
$$I = \int_0^1 \sqrt{1-x^2}dx.$$
   (a) Approximate I by hand using a 4-panel composite trapezoid method and calculate the absolute error. (Here ''panel'' refers to the number of times to apply the local rule within the composite rule.)
   (b) Approximate I by hand using a 2-panel composite Simpson method and calculate the absolute error.

2. Derivation of an integration rule:
   (a) Use the Lagrange interpolating polynomial to derive a local integration rule for approximating
$$\int_0^{3h} f(x)dx \tag{1}$$
   by evaluating $f(x)$ at ONLY the points 0, h, and $3h$.
   (b) Use Taylor series expansions to determine the truncation error of the method. Write out the lowest order error term explicitly, and use big-O notation for higher order error terms.
   (c) Use a change of variables or a symmetry argument to construct a local integration rule which only evaluates the function $f(x)$ at 0, $2h$, and $3h$.
   (d) Use Taylor series expansions to determine the truncation error of the method in part c. Write out the lowest order error term explicitly, and use big-O notation for higher order error terms.
   (e) Use Richardson extrapolation and the previous results to derive Simpson's 3/8'ths rule and a bound on the order of its truncation error.

3. The midpoint-rule is another integration rule that is a simple modification of Riemann's rule. Instead evaluating the height of a rectangle based on the left or right side, it evaluates the height in the middle of the rectangle.
$$\int_a^b f(x)dx \approx hf(\frac{a+b}{2}), \quad h := b-a.$$
   The composite form of the midpoint rule is
$$\int_a^b f(x)dx \approx h\sum_{i=0}^{n-1} f(x_i), \quad h := (b-a)/n, \quad x_i := a + \left(\frac{2i+1}{2}\right)h.$$
   (a) Calculate the truncation error of the regular midpoint rule as $h \to 0$.
   (b) Use your result to calculate the truncation error of the composite midpoint rule as $h \to 0$.
   (c) How does the midpoint rule perform, compared to our other integration methods?

(d) Use the composite midpoint rule with 5 panels to approximate

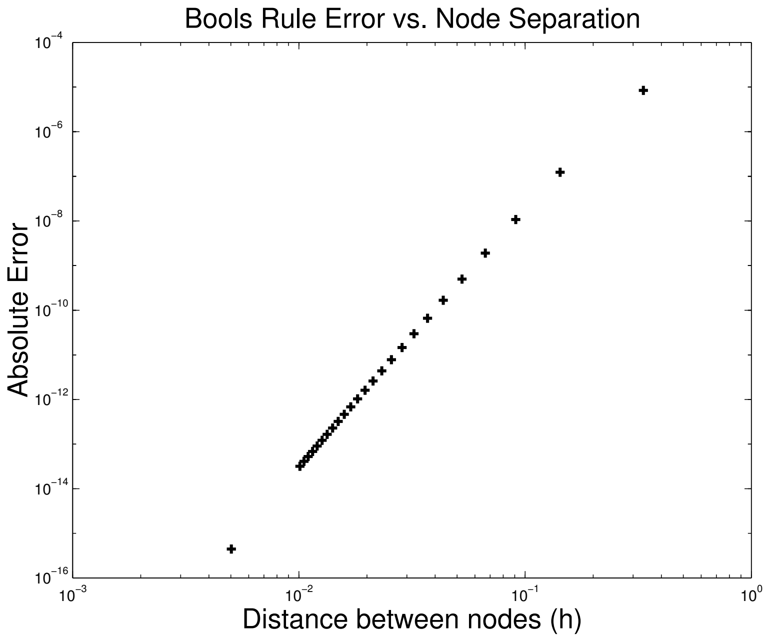
$$\int_{-1}^{1} \ln\left(1 - x^2 + \cos(\pi x/2)\right) dx$$

(e) Compare your result to the trapezoid method with 5 panels.

4. Using Boole's rule to integrate $\cos(x)$ from 0 to $\pi/2$, I obtained the following graph showing the error as the step size h decreases. If the error has the form

$$Ch^n,$$

use two points from this plot to estimate C and n. Round n to the nearest integer.



5. When calculating integrals numerically, we can sometimes get better results if we transform the domain to smooth out or compactify the integrand.

(a) Use the substitution $u = \sqrt{x}$ to remove the singularity from the integral

$$\int_{0}^{2} \frac{\cos(kx)}{\sqrt{x}(2 + x^2)} \mathrm{d}x$$

(b) Use the substitution $x = u/(1 - u)$ to transform the integral

$$\int_{0}^{\infty} \cos(x) e^{\sqrt{x} - x} \mathrm{d}x$$

to an integral from 0 to 1.

6. Consider the integral

$$J = \int_{0}^{3} (x + 1) \cos(e^x) dx$$

(a) Approximate J using the Riemann rule and 101 points.

(b) Approximate J using the trapezoid rule and 101 points.

(c) Approximate J using Simpson's rule and 101 points.

(d) (Bonus) Approximate J using Boole's rule and 101 points.