Homework 5, Math 451, Spring 2013

due Friday, February 22nd

This home covers material from the end of chapter 3 and the beginning of chapter 4 of Mathews and Fink.

1. Calculate the LU factorization of the following matrix by hand.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 6 \\ -5 & 2 & -1 \end{bmatrix}$$

2. While row operations leading to an LU factorization is the standard solution method linear systems, this choice was somewhat arbitrary. An UL factorization ($UL = A$, where L is lower triangular with ones on the diagonal and U is upper triangular) can be produced by column operations. Write a Matlab function "ul_factor(A)" which takes a square, non-singular matrix A as its argument and returns two matrices, $[U, L]$, such that U is upper triangular, L is lower-triangular with 1's on the diagonal, and $A = UL$. The first line of your function should read "function [U,L] = ul_factor(A)". You should calculate this factorization using column operations. Start at the bottom right corner of A and move up and left. The rules for column operation are just like the rules for row operations:

   - You can multiply a column by a scalar,
   - You can add two columns.

   Keep track of these operations in the L matrix as you make A upper triangular.

   Use your program to factor the following two matrices (the first, you can also do by hand to check for your calculations)

$$A = \begin{bmatrix} 4 & 3 & 4 \\ 5 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 22 & 10 & 8 & 15 & 3 \\ 19 & 19 & 11 & 5 & 0 \\ 22 & 16 & 11 & 12 & 2 \\ 9 & 6 & 8 & 8 & 1 \\ 3 & 0 & 2 & 5 & 1 \end{bmatrix}$$

   Also, please submit your code on Angel, in the appropriate box, as a single matlab function file. Be sure to remove all debugging outputs and disp() calls before you submit. Any extra output will be treated as an error.

3. Consider the linear equation $Mx = b$ for each of the following M's and b's. Calculate the rank of M in each case.

   (a)

$$M = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \quad b = [1, 2]^T$$

   (b)

$$M = \begin{bmatrix} 4 & 2 & 0 \\ -1 & 4 & 8 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad b = [1, 2, 3, 4]^T$$

(c)

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad b = [1,2,3]^T$$

(d)

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad b = [1,2,3]^T$$

(e)

$$M = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 0 \\ 4 & 0 & -2 \end{bmatrix}, \quad b = [1,2,3]^T$$

(f)

$$M = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}, \quad b = [1,2,3]^T$$

4. Determine for each of the matrices in problem 3(a)-3(f), whether the system $Mx = b$ has zero, one, or more than 1 solution.

5. Use Matlab's backslash operator to try to solve $x = M\backslash b$ for each of 3(a)-3(f), and compare Matlab answer with your expectations from problem 4.

6. In mathematical biology, the probability x that a newly introduced disease cases an epidemic can be calculated by solving a system of two coupled nonlinear equations like

$$u = e^{(u-1)a+(v-1)b}, \quad v = e^{(u-1)c+(v-1)d}$$

   This can be written as

$$\vec{x} = H(\vec{x}), \quad \text{with} \quad \vec{x} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \text{and} \quad H(\vec{x}) = \begin{bmatrix} e^{(u-1)a+(v-1)b} \\ e^{(u-1)c+(v-1)d} \end{bmatrix}.$$

   For our example, we will take $a = 2, b = 3, c = 1, d = 1$.

   (a) Calculate 3 steps of fixed-point iteration

$$\vec{x}_{t+1} = H(\vec{x}_t)$$

   starting with an initial guess of $\vec{x}_0 = (0,0)$ to find $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$.

   (b) Using $F(x) = x - H(x)$, calculate 3 steps of Newton--Raphson iteration

$$\left(\frac{dF}{d\vec{x}}\right)(\vec{x}_t - \vec{x}_{t+1}) = F(\vec{x}_t)$$

   starting with an initial guess of $\vec{x}_0 = (0,0)$ to find $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$. You should use Matlab's backslash operation to solve the linear system at each step.

7. Find the function $f_4(x)$ equal to the first four terms of the Taylor series approximation of $f(x) = e^{1/x}$ around $x = 1$. Include a plot of $f(x)$ and $f_4(x)$, clearly labeled, over an x-domain of $[0,2]$, and a y-range of $[0,6]$.

8. Draw the Newton Polygon for the curve

$$3x^3 + 5x + x^2y - 2y^2 - 8y = 4$$

   and find 2 asymptotic formulas for the curve's behavior when $|x| + |y|$ is large.