Math 450 Final Exam Review

1. Consider the sequence of rewritings

$$\begin{aligned}
0 :& a \\
1 :& b \\
2 :& ac \\
3 :& bab \\
4 :& acbac \\
5 :& babacbab \\
6 :& acbacbabacbac \\
7 :& babacbabacbacbabacbab
\end{aligned}$$

   (a) What are the rewrite rules for each iteration?

   (b) How many a's will there be in the 8th step?

2. What 5 dimensional parameters are used to predict the drag force on a ship? How many dimensionless groups does the Buckingham Pi theorem predict for these parameters? Please derive.

3. Give 3 other real-world examples of power-laws.

4. Rewrite the following nonlinear 3rd-order equation for $y(x)$ as a system of 3 first-order equations.

$$y''' + y'y'' - y = x$$

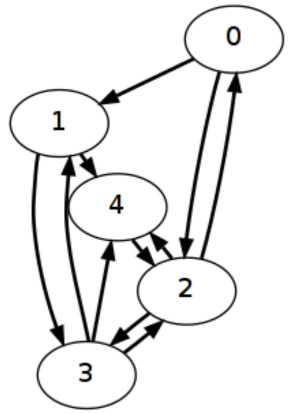
5. Draw a directed graph representing the discrete-time Markov chain for the count to a batter in baseball. How many states are in the chain?

6. Given a Markov chain $p(t+1) = Ap(t)$ for the weather in Oz, where state 1 is sunny, 2 is rainy, and 3 is snowy, and

$$A = \frac{1}{4}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix},$$

   if today is sunny, what is the probability that the weather will be sunny in 3 days?

7. Consider the directed graph below representing a continuous-time Markov chain where all the rates are 1 per day.



Find the steady-state distribution. If each state is numbered according to its payoff per day, what is the average payoff per day (note that the average payoff is independent of the initial state). Then set up a system of differential equations for the value as a function of time when there is a positive discount rate. What state is it best to start in? Does the best initial state change as the discount rate changes?

8. What are the steady-state solutions of the discrete logistic equation

$$x(t+1) = rx(t)(1 - x(t))?$$

When is each stable? What condition must a period-2 solution of the discrete logistic equation satisfy? Find an exact period-2 solution that is not a period-1 solution when $r = 10/3$.

9. Calculate the square error between the line $y = 2x - 1$ and the 3 data points $\{(-1, -1), (1, 0), (4, 1)\}$.

10. Suppose $p_n(t)$ is the probability of having n atoms of some isotope at time t. The continuous-time Markov process equations

$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = r(n+1)p_{n+1} - rnp_n$$

for the stochastic radioactive decay of the set of atoms. If there are initially m atoms, show that the solution is given by the binomial distribution

$$p_n(t) = \frac{m!}{n!(m-n)!} e^{-rnt}(1 - e^{-rt})^{m-n}$$

How is this related to the standard exponential decay equation $c(t) = c(0)e^{-rt}$?

11. What is the mathematical formulation of Newton's second law?

12. Give one derivation for the equation of motion of a catenary describing a hanging chain.

13. (Reading systems of equations) In the study of viral dynamics in the human body, $T(t)$ represents the concentration of target cells at time t, $I(t)$ represents the concentration of infected cells, and $V(t)$ represents the concentration of virus. Changes to these three state variables are governed by the system of differential equations

$$\dot{T} = \lambda - dT - kVT,$$
$$\dot{I} = kVT - \delta I,$$
$$\dot{V} = pI - cV.$$

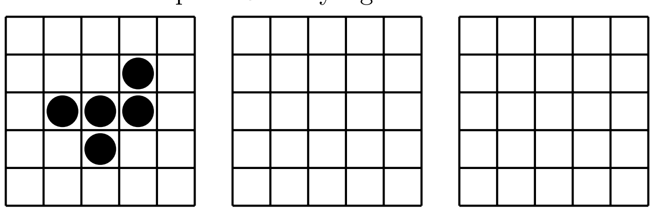
Find the parameter that matches each of the following.

(a) ______ The per-capita clearance rate of virus.
(b) ______ The rate of infection of target cells per cell per virus.
(c) ______ The rate of virus production by infected cells per cell.
(d) ______ The per-capita rate of death of infected cells.
(e) ______ The per-capita natural rate of death of target cells.
(f) ______ The production rate of target cells

14. In reality, most of the virus particles (99%) produced by an infected cell are defective and can not infect other cells. Change the system of equations above to include a 4th state variable $W(t)$ representing the concentration of defective virus, while $V(t)$ represents the concentration of infectious virus.

15. What is the Poisson distribution? How many parameters does it have and what are their interpretations? What assumptions is it based on, and why is it easier to calculate than a binomial distribution?

16. Iterate two steps of Conway's game of life from the initial condition given below.

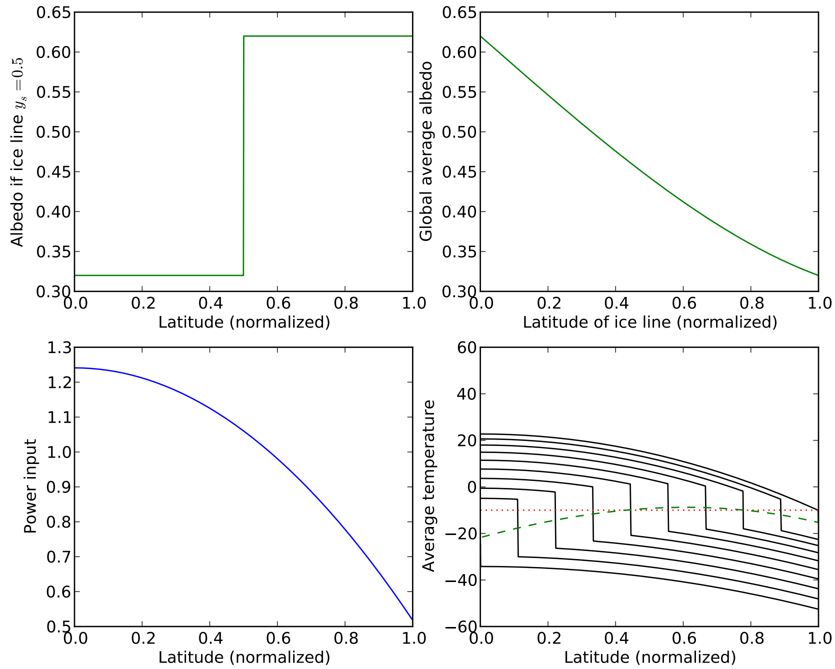


17. In the "snowball-earth" theory of global climate, why does albedo depend on global average temperature?

18. In class, we discussed Budyko's climate model

$$\overline{T}(y,t) = \int_0^1 T(y,t)dy$$
$$R\dot{\overline{T}} = Q(1 - \overline{\alpha}(y_s)) - (A + B\overline{T})$$
$$R\dot{T} = Qs(y)(1 - \alpha(y, y_s)) - (A + BT) + C\left[\overline{T} - T\right]$$
$$T(y_s) = 1/2\left[\lim_{y\to y_s^-} T(y) + \lim_{y\to y_s^+} T(y)\right]$$

leading the plots



What does this model conclude about the potential steady-states of the global climate? Explain how the equations lead to this conclusion.

19. In pseudocode, describe how we would go about fitting the viral dynamics model above to data on viral load in a patient (note that we can not observe target cells or infected cells, only virus).

20. Let $K(p)$ be the probability that there is a path from 1 side of a 2-d square (100x100) lattice to the other side under site-percolation with von Neumann neighbors (4) when a fraction p of the sites are blocked. Sketch $K(p)$. Sketch a second version of the curve under Moore neighbors (8) on the same plot.

21. Explain the phase-translation in the percolation model that occurs as more sites are filled in, blocking flow.

22. What is the Stefan--Boltzmann law? What's the predicted average temperature of Venus?