This homework covers material on scaling laws and Markov chains.

1. In class, we discussed one form of the preferential attachment model. Another important form is the citation patterns found in scientific articles, as described in section 2.9 of Tung. Rather than doing equations, let’s make a simulation of a simplified version of this.

   (a) Start with a list containing the integers 1 and 2.
   (b) Choose one number randomly from the list and append it back to the list. You can use the “sample()” function from the “random” module to do this.
   (c) Append the next number (3) to the list.
   (d) Repeat these last two steps 1,000 times, incrementing the new number each time until you get a list of 2002 integers.

   Now, count up how many times each integer appears in the list and plot a rank-value plot on a log-log scale. Explain how we know that the plot exhibits a power-law scaling, and approximate the scaling exponent.

2. The following examples are to give you some practice with Markov chain calculations. For each transition matrix, draw a directed graph, with nodes labeled for each state and edges labeled with their transition probabilities. Then, try to find the equilibrium distribution \( \tilde{p} \) such that \( \tilde{p} = A\tilde{p} \) and discuss the meaning of your result.

   (a) 
   \[
   A := \begin{bmatrix}
   \frac{1}{2} & 0 & \frac{1}{3} \\
   \frac{1}{2} & \frac{1}{3} & 0 \\
   0 & \frac{2}{3} & \frac{1}{2}
   \end{bmatrix}.
   \]

   (b) 
   \[
   A := \begin{bmatrix}
   1 & \frac{1}{2} & 0 & 0 \\
   0 & \frac{1}{2} & \frac{1}{2} & 0 \\
   0 & \frac{1}{4} & \frac{1}{2} & 0 \\
   0 & 0 & \frac{1}{4} & 1
   \end{bmatrix}.
   \]

3. The breakdown of machines and structures are often modelled using Markov chains. One example of this is the deterioration of bridges. Imagine a new rope bridge in the Andes mountains held up by 3 jut ropes. Every time a person crosses the bridge, there is a chance that one of the ropes will break. Let \( p_n \) be the probability that a rope breaks when there are \( n \) ropes, and assume that two ropes never break at once. For our purposes, \( p_n = \frac{1}{200n} \).

   (a) Draw graph of this Markov chain. Label all nodes with their corresponding state, and all edges with their transition probability.
   (b) Construct a transition matrix for changes in the bridges state every time a person crosses the bridge.
   (c) If the Markov chain begins when the bridge is first built, what is the initial condition?
   (d) What is the probability that the bridge will have collapsed by the time 1000 people have crossed it? (Hint: Use python to perform this calculation. The function \texttt{linalg.matrix.power} will be useful. Be sure to explain your answer.)

4. Chapter 2.10, # 6 from Tung.