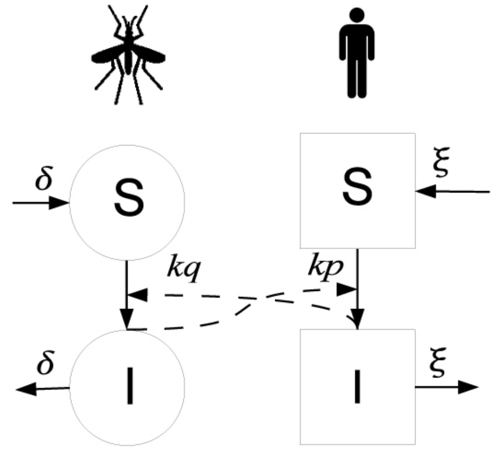
Math 450 in-class worksheet practice for compartmental models.

1. The diagram at right was published as a description of the famous Ross-Macdonald model of malaria transmission between susceptible and infected people and mosquitoes.



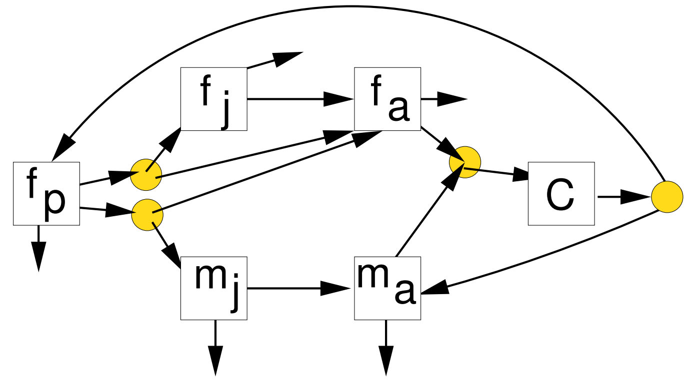
   (a) Fix the variable labels so that all 4 are different.

   (b) What do you think are the differences represented by the dashed verses solid arrows?

   (c) Using your fixed variable labels, list the 6 reactions, with rate labels, that form the reaction network equivalent to this diagram.

   (d) Write down the system of four differential equations equivalent to our reaction network.

2. The North-Atlantic right whale is a species of baleen whale that can grow to 15 meters in length and 70 megagrams in mass. Right whales were heavily hunted in the 19th century, and today only around 500 survive. Here is a hypergraph representing a simple compartmental model of the life-cycle of right whales. Models like this are important in estimating minimum viable population sizes.



(a) How many variables are there in this life-cycle model? ________

(b) Label each reaction in the diagram with a parameter.
How many parameters are there? ________

(c) Guess at a plain-English description of each state. (Hint: first guess what each of the 4 binary reactions represents)

(d) Write down the system of differential equations for the state variables represented by the hypergraph using your parameters.

3. In a 2015 manuscript, Silva, Rosa, Alves, and Carvalho propose a model for the relationship between marketing and customer recruitment based on the compartmental model given by the equations below. In these equations, $R(t)$ is the number of referral customers in time t, C is the number of regular customers, P_R is the number of potential referral customers, and P_C is the number of potential regular customers. (Here, the prime's denote time derivatives)

$$\begin{cases} C' = \lambda_7 R - (\varepsilon + \beta_1 + \lambda_5)C + (\lambda_1 + m\lambda_4)P_C + \lambda_2 R P_C \\ R' = \lambda_5 C - (\varepsilon + \beta_2 + \lambda_7)R + (\lambda_3 + m\lambda_4)P_R + (\lambda_2 + m_R\lambda_6)RP_R \\ P_C' = (1-\alpha)\gamma + \beta_1 C + \lambda_7 P_R - (\varepsilon + \lambda_5 + \lambda_1 + m\lambda_4)P_C - \lambda_2 R P_C \\ P_R' = \alpha\gamma + \beta_2 R + \lambda_5 P_C - (\varepsilon + \lambda_7 + \lambda_3 + m\lambda_4)P_R - (\lambda_2 + m_R\lambda_6)RP_R \end{cases}$$

Follow these steps to construction the equivalent reaction network. You might find it handy to cross out the terms as you use them.

(a) Find all binary reactions. (In this model, all binary reactions can be written in a form the preserves the total number of people.)

(b) Find all source reactions that introduce new people into the system.

(c) Find all linear transition reactions that move people from one state to another.

(d) Find all reactions that remove people from the system.

(e) Draw a hypergraph equivalent to this set of reactions.

4. In 2009, Munz, Hudea, Imad, and Smith? proposed the following compartmental model as a theoretical description for a zombie outbreak. Let S be the number of susceptible (living) people, Z be the number of zombies, and let D be the number of dead people. Then

$$\begin{aligned}\dot{S} &= \rho S - \beta S Z - \delta S \\ \dot{Z} &= \beta S Z + \zeta D - \alpha S Z \\ \dot{D} &= \delta S + \alpha S Z - \zeta D.\end{aligned}$$

(a) Translate this model into a reaction network.

(b) Draw a hypergraph representation of this model.

(c) Give a simple explanation for what each reaction could be representing.

(d) According to this theory, is it possible for the "living" to "win"? Why? Do you agree with the model's predictions?

(e) Develop an improved compartmental model hypergraph of the zombie epidemic. Translate the improved model into a system of nonlinear differential equations.

(f) Plan an "intervention" (e.g. quarantine, vaccine, other) to prevent or minimize the impact of the zombie plague.

(g) Extend your model in (e) to include the effects of this intervention. Provide the hypergraph, and the equivalent system of nonlinear differential equations.