Congruence and symmetry groups

In class, we learned how, when order matters, we can count different possible objects by determining all the equivalent configurations based on allowed transformations of one example. Here are some practice problems.

- In class, we showed that in flatland, there are two different ways to color a triangle's corners. Now, let's do squares.
  - How many configurations are there for a square with 4 differently coloredcorners? Answer: $4! = 24$
  - How many different transformations of a given square are allowed in flatland? Answer: 4
  - How many different square colorings are there in flatland? List them all. Answer: $24/4 = 6$
- Now, what about other regular polygons in flatland?
  - How many ways are there to color a pentagon's 5 corners in flatland with 5 different colors? Answer: *There are $5! = 120$ pentagon coloring when we can not move them. In flatland, each pentagon has 5 rotation positions, so $5!/5 = 4! = 24$ different pentagons.*
  - How many ways are there to color an n-gon's corners in flatland? Find a formula. Answer: *By extrapolation, we might expect $n!/n = (n-1)!$ different n-gon colorings, up to symmetry transformations in flatland.*
- In sphereland, were we live, we can, of course, move things in 3 dimensions.
  - How many ways are there to color a square's 4 corners in sphereland? Answer: *In sphereland, there are only 3 different squares. One corner must be red, and then there are 3 choices for the color opposite red. The choices for the remaining corners don't matter because you can flip back and forth between them. The other way to see this is to observe that the symmetry group of flips and rotations of the square is the dihedral group of order 4, D_4, which has 8 elements, and $24/8 = 3$.*
  - How many ways are there to color a pentagon's 5 corners in sphereland? List them all. Answer: *The symmetry group of a regular pentagon is the dihedral group of order 5, D_5, which has 10 elements created by flipping and rotating the pentagon. Thus, $120/10 = 12$ different pentagons.*
  - How many ways are there to color an n-gon's corners in sphereland? Answer: *In general, $n!/(2n) = (n-1)!/2$.*
- How many ways are there to color the corners of a rectangle in sphereland? Answer: 6. *One corner is green. Then you have 3 choices for the neighboring corner on the long edge, and 2 choices for the neighboring corner on the short edge, and 1 for the last corner, so $3 \times 2 = 6$ different rectangles. Alternatively, the symmetry group of a rectangle is the Klein-4 group, K_4, and $|K_4| = 4$. Since there are 24 color configurations, $24/4 = 6$.*
- How many different ways are there to color the corners of a brick in sphereland? (a brick being a solid with 6 faces and 8 corners (vertices), all faces at right-angles to each other, but the length, width, and height all different) Answer: *This requires some 3-d thinking. Given a brick (like your textbook), there are only 4 symmetric configurations for placing it on your desk (2 flips, and 2 rotations for each flip). Since there are 8 corners, then there are $8!/4 = 10080$ colorings.*