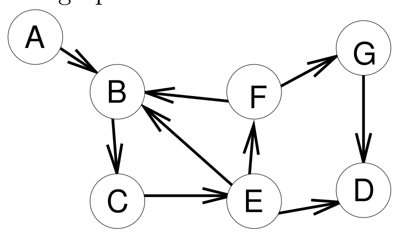
Homework, Math 311w

Extra problems for section 2.3: Relations

1. In mathematics, we often start with one simple object, and then use that object to build more complicated things. Consider the digraph below with node set $N = \{A, B, C, D, E, F, G\}$.

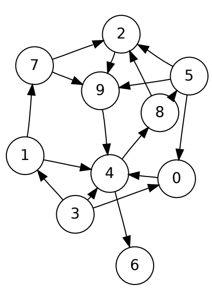


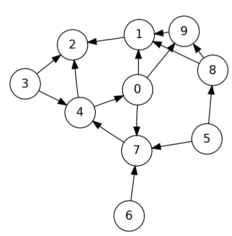
   - Define the relation $\succ$ on the set of nodes N so that $x \succ y$ when there is an edge going from x to y in the digraph.
   - The relation $\gg$ on the set of nodes N is such that $x \gg y$ if there is a set of nodes (possibly empty) $a_1 \ldots a_n$ such that $x \succ a_1$, $a_1 \succ a_2 \ldots a_n \succ y$. This means that there is a path (made of one or more edges, where all the edges are followed in the direction of the arrow) from x to y in the graph.
   - The relation $\equiv$ on the set of nodes N is such that $x \equiv y$ if x and y are the same or both $x \gg y$ and $y \gg x$ hold.

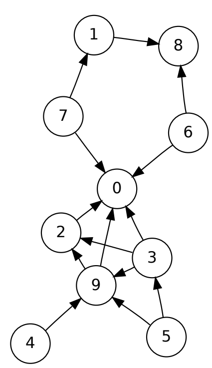
   (a) Draw adjacency tables for each of these 3 relations.(see the book's example)

   (b) Show that $x \succ y$ is antisymmetric, that $x \gg y$ is transitive, and that $x \equiv y$ is an equivalence relation.

2. For more practice, do the same analysis with the following directed graphs that define possible relations on the integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}$.







Answers

$x \succ y$	A	B	C	D	E	F	G
A	0	1	0	0	0	0	0
B	0	0	1	0	0	0	0
C	0	0	0	0	1	0	0
D	0	0	0	0	0	0	0
E	0	1	0	1	0	1	0
F	0	1	0	0	0	0	1
G	0	0	0	1	0	0	0

$x \gg y$	A	B	C	D	E	F	G
A	0	1	1	1	1	1	1
B	0	0	1	1	1	1	1
C	0	1	1	1	1	1	1
D	0	0	0	0	0	0	0
E	0	1	1	1	1	1	1
F	0	1	1	1	1	1	1
G	0	0	0	1	0	0	0

$x \equiv y$	A	B	C	D	E	F	G
A	1	0	0	0	0	0	0
B	0	1	1	0	1	1	0
C	0	1	1	0	1	1	0
D	0	0	0	1	0	0	0
E	0	1	1	0	1	1	0
F	0	1	1	0	1	1	0
G	0	0	0	0	0	0	1

$x \succ y$ is antisymmetric because whenever $x \succ y$ holds, $y \succ x$ does not hold. Specifically, no entry (marked with a $-$) opposite each T in the first table is in the relation, and there are no T's on the main diagonal. (the ''main diagonal'' starts at the top left corner and ends at the bottom right corner)

The second relation $x \gg y$ holds whenever there is a path, following arrows in the forward direction, from x to y. The relation is transitive because if there is a path from x to y and a path from y to z, then naturally we can string the two togethor to get a path from x to z.

The third relation $x \equiv y$ is reflexive (the main diagonal is filled with T's) and symmetric (it's the same if we flip the table across the main diagonal). Transitivity requires more careful argument. Assume $x \equiv y$ and $y \equiv z$. If $x = y$, then $x \equiv z$ by substitution. If $y = z$, the $x \equiv z$ by substitution. These cases need to be handled separately because $\gg$ is not reflexive. Now, if $x \neq y$ and $y \neq z$, then $x \equiv y$ and $y \equiv z$ means there are paths from x to y, y to z, z to y, and y to x. Joining these paths together, we get paths from x to z and z to x, so $x \gg z$ and $z \gg x$. Thus, $x \equiv z$ when $x \neq y$ and $y \neq z$. This covers all the cases possible, and we conclude $\equiv$ is transitive. Since $\equiv$ is reflexive, symmetric, and transitive, this definition of $\equiv$ is an equivalence relation.