Math 311, Quiz 7

Name:____

Instructions: Clearly answer each of the questions below. Remember to check the back side – if blank, you can use it for scrap work. Use full sentences and proper grammar. Show your work and any formulas you employ. Simplify all answers as far as possible. Box your answers.

1. What is the definition of a subgroup?

   Answer:*A subgroup is a subset of a group that is itself a group with respect to the given operation.*

2. The set $\{[0]_8, [1]_8, [4]_8\}$ is not a subgroup of the group $\mathbb{Z}_8$ under addition. Why?

   Answer: *This set is not a group because it is not closed under addition. For example,* $[1]_8 + [4]_8 = [5]_8$*, but* $[5]_8$ *is not in the given set, so it is not closed under addition. The set is also missing an additive inverses of* $[1]_8$*, which is another reason it is not a group.*

3. Find all the cyclic subgroups generated by elements of $\mathbb{Z}_8$ under addition and draw their Hasse diagram.

   Answer: *There are 4 different cyclicly generated groups in* $\mathbb{Z}_8$*, one for each factor of* 8*.*

$$\text{a subgroup equal to the full group,}\quad \langle [1]_8 \rangle = \langle [3]_8 \rangle = \langle [5]_8 \rangle = \langle [7]_8 \rangle = \{[0]_8, [1]_8, [2]_8, [3]_8, [4]_8, [5]_8, [6]_8, [7]_8\}.$$
$$\text{a subgroup of order 4,}\quad \langle [2]_8 \rangle = \langle [6]_8 \rangle = \{[0]_8, [2]_8, [4]_8, [6]_8\}$$
$$\text{a subgroup of order 2,}\quad \langle [4]_8 \rangle = \{[0]_8, [4]_8\}$$
$$\text{a subgroup of order 1,}\quad \langle [0]_8 \rangle = \{[0]_8\}$$

$$\begin{tikzcd}
\langle [1]_8 \rangle = (\mathbb{Z}_8, +) \arrow[d] \\
\langle [2]_8 \rangle \arrow[d] \\
\langle [4]_8 \rangle \arrow[d] \\
\langle [0]_8 \rangle
\end{tikzcd}$$

4. The set of integers $\mathbb{Z}$ is an infinite group under addition. The positive integers $\mathbb{N}\backslash\{0\} = \{1, 2, 3, 4...\}$ are a closed subset of $\mathbb{Z}$. Why do the positive integers not form a subgroup of $\mathbb{Z}$ under the addition operation?

   Answer: *The positive integers do not form a group under addition because there is no identity and there are no inverses. The identity element should be* 0, *but* $0 \notin \mathbb{N}\backslash\{0\}$. *The additive inverse of* 2 *is* -2, *but* $-2 \notin \mathbb{N}\backslash\{0\}$.

5. State Lagrange's theorem.

   Answer: *Given any finite group* G *and any subgroup* H *of* G, *the order of* H *evenly divides the order of* G.

6. Consider $\mathbb{Z}_{15}^{\times}$, the group of invertible congruence classes modulo 15 under multiplication.

   (a) List all the elements of $\mathbb{Z}_{15}^{\times}$.
   Answer:
   $$\mathbb{Z}_{15}^{\times} = \{[1]_{15}, [2]_{15}, [4]_{15}, [7]_{15}, [8]_{15}, [11]_{15}, [13]_{15}, [14]_{15}\}$$

   (b) Find all the left cosets aH for the subgroup $H = \{[1]_{15}, [4]_{15}\}$ of $\mathbb{Z}_{15}^{\times}$.
   Answer: *The group is order 8, and the subgroup is order 2, so there will be* $8/2 = 4$ *cosets* aH.

   $$\{[1]_{15}, [4]_{15}\}$$
   $$\{[2]_{15}, [8]_{15}\}$$
   $$\{[7]_{15}, [13]_{15}\}$$
   $$\{[11]_{15}, [14]_{15}\}$$