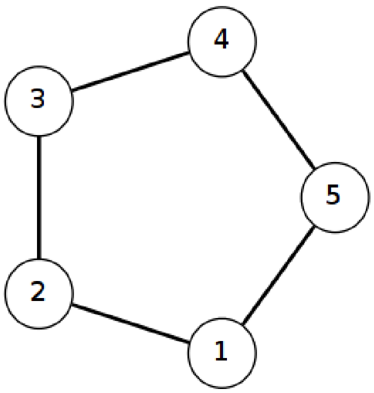
Math 311, Quiz 7

Name:________________________________

Instructions: Clearly answer each of the questions below. Remember to check the back side – if blank, you can use it for scrap work. Use full sentences and proper grammar. Show your work and any formulas you employ. Simplify all answers as far as possible. Box your answers when appropriate. In the problems below, assume that set G forms a group under the operation $*$.

1. Considering the pentagon below, determine which permutations represent symmetry transformations in **flatland** (2 dimensions), **sphereland** (3 dimensions), or are **not a symmetry** transformation of the labelled pentagon. Note that everything that is a symmetry in 2 dimensions is also a symmetry in 3 dimensions. Give the most useful answer.



(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ Answer:*flatland*

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix}$ Answer:*not a symmetry*

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ Answer:*sphereland*

(d) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 5 & 1 \end{pmatrix}$ Answer:*not a symmetry*

(e) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$ Answer:*flatland*

(f) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$ Answer:*flatland*

2. For $a, b \in G$, what is the inverse of $a * b * a^{-1} * b^{-1}$?

Answer:$(a * b * a^{-1} * b^{-1})^{-1} = (b^{-1})^{-1})(a^{-1})^{-1}b^{-1}a^{-1} = bab^{-1}a^{-1}$

3. What is the definition of a subgroup?

   Answer:*A subgroup is a subset of a group that is itself a group with respect to the given operation.*

4. The set $\{[0]_8, [1]_8, [4]_8\}$ is not a subgroup of the group $\mathbb{Z}_8$ under addition. Why?

   Answer: *This set is not a group because it is not closed under addition. For example,* $[1]_8 + [4]_8 = [5]_8$*, but* $[5]_8$ *is not in the given set, so it is not closed under addition. The set is also missing an additive inverses of* $[1]_8$*, which is another reason it is not a group.*

5. The group of quaternion Q_4 was discovered as a generalization of complex numbers, and plays a useful role in the study of electromagnetism. It is generated by four elements $\{n, i, j, k\}$, with the simplifying equivalences $n = i^2 = j^2 = k^2$, $n^2 = e$, $ni = in$, $nj = jn$, $nk = kn$, and $ijk = n$, where e is the identity element. For each of the words below, use the equivalences above to reduce the word to the form $n^\alpha x$ where $\alpha \in \{0, 1\}$ and $x \in \{e, i, j, k\}$.

   (a) ijn Answer:*Since* $n = k^2$, $ijn = ijk^2 = (ijk)k = nk$

   (b) kji Answer: *Using our previous answer,* $kji = kjie = kjin^2 = k(jin)n = k(nk)n = n^2k^2 = en = n = ne$

6. If $g \in G$, prove that the order of g is not greater than the number of elements in G.

   Answer: *The order of* g *is the smallest positive integer for which* $g^m = e$*. Let* S *be the set of all powers of* g*.*

   $$S = \{e, g, g * g, g * g * g, ...\} = \{g^0, g^1, g^2, g^3, ...\}.$$

   Then the order of g *will equal the number of distinct elements in* S *because the sequence will repeat at that period. Now, we'll do a proof by contradiction. Assume* S *has more elements than* G*. Then by the pigeon-hole principle, there must be at least one* $x \in S$ *such that* $x \notin G$*. But since* $x \in S$*,* $x = g^m$ *for some* m*. Since* $g \in G$ *and a group is always closed under its operation,* $x \in G$*. This is a contraction. Thus, it must be that* S *never has more elements than* G*. Thus, the order of* g *must not be greater than the number of elements in* G*.*