Name:______________________________________

Instructions: Clearly answer each of the questions below. Remember to check the back side – if blank, you can use it for scrap work. Use full sentences and proper grammar. Show your work and any formulas you employ. Simplify all answers as far as possible. Box your answers.

1. Calculate the standard form of $[18]_{21} + [5]_{21}[8]_{21}$.

   Answer: $[18]_{21} + [5]_{21}[8]_{21} = [18]_{21} + [40]_{21} = [58]_{21} = [2 \times 21 + 16]_{21} = [16]_{21}$

2. Prove that a congruence class can not be both invertible and a zero-divisor.

   Answer: *We will argue by contradicting the opposite of the theorem. Assume a congruence class* $[a]_n$ *has both an inverse* $[i]_n$ *and a zero-divisor* $[z]_n \neq [0]_n$*. Then according to the definition of inverses,* $[a]_n[i]_n = [1]_n$*. And according to the definition of zero-divisors,* $[a]_n[z]_n = [0]_n$*. If we multiply all three together,*

   $$[z]_n = [z]_n[1]_n = [z]_n([a]_n[i]_n) = ([z]_n[a]_n)[i]_n = [0]_n[i]_n = [0]_n.$$

   However, this contradicts our assumption that $[z]_n \neq [0]_n$*. Thus, it is impossible to have both an inverse and a zero divisor.*

3. Find $[21]_{34}^{-1}$. Answer:

$$\left(\begin{array}{cc|c} 1 & 0 & 21 \\ 0 & 1 & 34 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 21 \\ -1 & 1 & 13 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & 8 \\ -1 & 1 & 13 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & 8 \\ -3 & 2 & 5 \end{array}\right) \to$$
$$\left(\begin{array}{cc|c} 5 & -3 & 3 \\ -3 & 2 & 5 \end{array}\right) \to \left(\begin{array}{cc|c} 5 & -3 & 3 \\ -8 & 5 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 13 & -8 & 1 \\ -8 & 5 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 13 & -8 & 1 \\ -34 & 21 & 0 \end{array}\right)$$

$13(21) - 8(34) = 1$, *or* $(13)(21) = 1 + 8(34)$, *so* $[21]_{34}^{-1} = [13]_{34}$.

4. Find all integers x such that $x \equiv_9 7$ and $x \equiv_{11} 2$.

   Answer: *From Sunzi's remainder theorem, if* $x \equiv_{n_1} b_1$ *and* $x \equiv_{n_2} b_2$*, and* $n_1 r_1 + n_2 r_2 = 1$*,*

   $$x \in [b_2 n_1 r_1 + b_1 n_2 r_2]_{n_1 n_2}.$$

   There, $9 \times 5 + 11 \times (-4) = 1$*, so*

   $$x \in [2 \times 9 \times 5 - 7 \times 11 \times 4]_{9\times 11} = [79]_{99},$$

   or said differently, $x \in 79 + 99\mathbb{Z}$*.*

5. Are there any integers x, y, and z such that $x^2 + y^2 + z^2 \equiv_8 7$? Explain.

   Answer: *There are no integers that solve this congruence equation. It is easy to check that for any integer* x*,* $x^2 \equiv_8 0$*,* $x^2 \equiv_8 1$*, or* $x^2 \equiv_8 4$*. There is no way to add 3 of these together so that* $x^2 + y^2 + z^2 \equiv_8 7$*. There are* 10 *possibilities.*

   $$\begin{gathered}
   0+0+0 \equiv_8 0 \\
   0+0+1 \equiv_8 1 \\
   0+1+1 \equiv_8 2 \\
   1+1+1 \equiv_8 3 \\
   1+1+4 \equiv_8 6 \\
   1+4+4 \equiv_8 1 \\
   4+4+4 \equiv_8 4 \\
   0+4+4 \equiv_8 0 \\
   0+0+4 \equiv_8 4 \\
   0+1+4 \equiv_8 3
   \end{gathered}$$