Name:____________________________________

Instructions: Clearly answer each of the questions below. Remember to check the back side – if blank, you can use it for scrap work. Use full sentences and proper grammar. Show your work and any formulas you employ. Simplify all answers as far as possible. Box your answers when appropriate.

1. State the binomial theorem.

   Answer: *The binomial theorem says that for any positive integer n, and any real number x,*

$$(1+x)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k.$$

2. What do we mean when we say that two numbers are relatively prime?

   Answer: *Two numbers are relatively prime when there is no integer greater than 1 that evenly divides both.*

3. Prove that for all positive integers n, $3|5^n - 2^n$.

   Answer: *I will prove this to you using regular induction. The smallest positive integer is* 1. *If $n = 1$, then $5^n - 2^n = 5 - 2 = 3$ and $3|3$, so for our base case, if $n = 1$, then $3|5^n - 2^n$. Now, let's assume for some integer k, $3|5^k - 2^k$. Then, when $n = k + 1$,*

$$\begin{aligned} (5^n - 2^n)|_{n=k+1} &= 5^{k+1} - 2^{k+1} \\ &= 5(5^k) - 2(2^k) \\ &= 3(5^k) + 2(5^k - 2^k). \end{aligned}$$

   from the definition of divisibility, $3|3$. And we assumed $3|5^k - 2^k$. So, 3 must evenly divide any integer linear combination of 3 and $5^k - 2^k$, according to the linear combination theorem. In particular, $3|3(5^k) + 2(5^k - 2^k)$ and by arithmetic, $3|5^{k+1} - 2^{k+1}$. Since both the base case and the induction hypothesis are true, we conclude by the induction principle, that $3|5^n - 2^n$ for all positive integers n.

4. Find $\gcd(5^3 \times 31^4 \times 43,\ 7^3 \times 10 \times 31^8)$. (you don't need to multiply the answer out)

Answer: *We use the prime-factorization GCD theorem. Since* 10 *is not a prime, we transform to it's prime factorization first.*

$$\begin{aligned}\gcd(5^3 \times 31^4 \times 43,\ 7^3 \times 10 \times 31^8) &= \gcd(5^3 \times 31^4 \times 43,\ 2 \times 5 \times 7^3 \times 31^8) \\ &= 2^{\min(0,1)}5^{\min(3,1)}7^{\min(0,3)}31^{\min(4,8)}43^{\min(1,0)} \\ &= 2^0 5^1 7^0 31^4 43^0 \\ &= 5 \times 31^4.\end{aligned}$$

done.

5. Prove the relative-prime divisibility theorem from class that if $\gcd(a, b) = 1$ and $a|bc$, then $a|c$.

Answer: *Suppose a, b, and c are natural numbers for which $\gcd(a, b) = 1$ and $a|bc$. Then, by the duality theorem, there are integers s and t so that $1 = as + bt$. If we multiply both sides by c, then $c = acs + bct$. Now, we can see from inspection that $a|ac$ and by our initial supposition, $a|bc$. Then, by the linear combination theorem from class, a must evenly divide any integer linear combination of ac and bc, including $acs + bct$. Then, by substitution, $a|c$. Q.E.D.*