Math 311 practice

1. What is the definition of order of a congruence class?

2. If p and q are prime, what is the general formula for $\phi(p^3q)$?

3. Calculate each of the following.

   (a) The totient of 37.

   (b) The order of $[5]_{12}$.

   (c) The order of $[9]_{15}$.

4. Find $[11]_{201}^{-1}$.

5. State the Division Theorem

6. Use a counterexample to disprove the conjecture

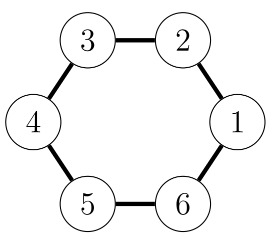
   "For any positive integers $n = pq$, $\phi(n) = \phi(p)\phi(q)$."

7. Give an example of a theorem that can be proven using the induction principle.

8. Translate the following sentence into propositional logic, using p to represent "$\gcd(a,b) = 1$", q to represent "$a|bc$", and r to represent "$a|c$".

   The statement "If $\gcd(a,b) = 1$ and $a|bc$, then $a|c$" is logically equivalent to the statement "if $\gcd(a,b) = 1$, then $a|bc$ implies $a|c$."

9. Consider the hexagon with labelled nodes

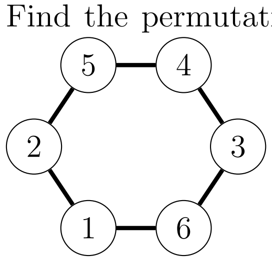


$$\begin{pmatrix}1&2&3&4&5&6\\1&6&3&4&5&2\end{pmatrix} \quad \begin{pmatrix}1&2&3&4&5&6\\6&5&4&3&2&1\end{pmatrix} \quad \begin{pmatrix}1&2&3&4&5&6\\3&4&5&6&1&2\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4&5&6\\6&1&2&3&4&5\end{pmatrix} \quad \begin{pmatrix}1&2&3&4&5&6\\6&4&5&2&3&1\end{pmatrix} \quad \begin{pmatrix}1&2&3&4&5&6\\5&4&3&2&1&6\end{pmatrix}$$

   (a) Cross-out each of the permutations above that is **NOT** a symmetry of the given hexagon.

   (b) Find the permutation f representing the transformation of the hexagon above into



   (c) Calculate the inverse permutation f^{-1}.

10. Use a truth-table to determine if $(\neg x \wedge (y \to x))$ is a tautology, contradiction, or contingency.

11. State the definition of a partial ordering, and give an example of a relation that satisfies this definition.

12. Why is $x|y$ not an equivalence relation?

13. Euclid's algorithm

    (a) State the theorem that Euclid's algorithm is based on?

    (b) Use Euclid's algorithm to calculate $\gcd(184, 115)$.

14. Find all the integer solutions x of each linear congruence relation.

    (a) $14x \equiv_{35} 21$.

    (b) $10x \equiv_{25} 3$.

15. Show that if $\gcd(m, 96) = 1$, then $m^{35} - m^3 \equiv_{96} 0$.

16. Find a truth-assignment for x, y, and z for which the following contingency is true.

$$(\neg z \wedge (x \vee \neg y)) \wedge (z \vee (x \wedge y))$$

17. "There are infinitely many prime numbers."

    (a) What method is used to prove this theorem?

    (b) Provide a proof.

18. Given that S_3 is the set of permutations of the set $\{1, 2, 3\}$,

$$r = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

    list the cosets of S_3 under $\langle r \rangle$.

19. Complete the Cayley table for a group $(\{0, 1, 2\}, \odot)$.

$\odot$	2	0	1
2			
0			1
1	0		

    Which element is the identity element?

20. Complete the Cayley table for a group $(\{0, 1, 2, 3, 4, 5, 6, 7\}, \boxdot)$. (updated 12-12)

$\boxdot$	5	0	7	2	1	6	3	4
5	0		1			3		
0		4	6	5			1	
7	3	6	0	1				
2	4	5	3	0	7	1	6	
1	7	3	2		0		4	1
6	1			3	2	0		
3	6	1					0	
4								

    Which element is the identity element?

21. Fill in the missing entries in the group multiplication table below.

	1	a	b	c	d	e	f	g
1	1							
a	a	1	c	b	e	d	g	f
b	b	c	a		g	f	d	e
c	c	b	1		f	g	e	d
d	d	e	f	g	a	1	c	
e	e	d	g	f	1	a	b	c
f	f	g		d	b	c	a	1
g	g	f	d	e	c	b	1	a

22. Calculate $\gcd(18, 24)$.

23. Prove $7|12^n - 5^n$ for all positive integers n.

24. Determine the prime factorization of 396.

25. Find all solutions x for each of the following equations.

    (a) $15 + x \equiv_{18} 4$

    (b) $8x \equiv_{15} 7$

    (c) $18x \equiv_{24} 10$

26. If $Q = \{a, b\}$ and $R = \{-2, 13, 18\}$, Evaluate

$$(Q \times R) \cap \{(a, -2), (-2, a), (c, -2), 13, (a, c), (b, 18)\}.$$

27. Prove $X \cap X^c = \emptyset$.

    Two sets are equal if and only if being in one is always equivalent to being in the other.

$$\text{Given} \quad (X \cap X^c = \emptyset)$$
$$\leftrightarrow \forall a, ((a \in (X \cap X^c)) \leftrightarrow a \in \emptyset) \quad \text{by def. of set equality}$$
$$\leftrightarrow \forall a, ((a \in X \wedge a \in X^c) \leftrightarrow a \in \emptyset) \quad \text{by def. of intersection}$$
$$\leftrightarrow \forall a, ((a \in X \wedge \neg(a \in X)) \leftrightarrow a \in \emptyset) \quad \text{by def. of set complement}$$
$$\leftrightarrow \forall a, (F \leftrightarrow a \in \emptyset) \quad \text{by logical consistency}$$
$$\leftrightarrow \forall a, (F \leftrightarrow F) \quad \text{by def of the empty set}$$
$$\leftrightarrow \forall a, T \quad \text{by def of logical equivalence}$$
$$\text{True}$$

28. How many different subsets of the set $\{a, b, c, d, e\}$ are there?

29. Construct a truth-table for $(p \wedge q) \to (q \to p)$.

30. What is an equivalence relation?

31. Suppose we have two permutations

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

(a) Find a^{-1}.

(b) Solve $a@x = b$ for x.

32. Find the order of each of the following under the given operation:

    (a) The permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$ under function composition.

    (b) The congruence class $[3]_7$ under multiplication.

    (c) The congruence class $[8]_{52}$ under addition.

33. The set $\{True, False\}$ is a group under the binary operation $a * b := (a \vee b) \wedge \neg(a \wedge b)$. Construct the group multiplication table.

34. Given the cyclic group $(\mathbb{Z}_8, +)$ and the subset $H = \{[0]_8, [4]_8\}$

    (a) Show H is a subgroup of $\mathbb{Z}_8$.

    (b) List the right cosets of $\mathbb{Z}_8$ under H.

35. Construct the subgroup digraph of $(\mathbb{Z}_{225}, +)$. Determine which subgroup in the graph is the same as $\langle [114]_{225} \rangle$.

36. Why are there no solutions to the equation $3x^3 = y^3$?

37. What is the order of the direct-product group $\mathbb{Z}_4 \times S_3$?

38. If H is a subgroup of G and elements a and b of G are in the same coset of H, in what sense are a and b equivalent?

39. Translate the following mathematical propositions into plain English.

$$\forall x \in \mathbb{N} \exists y \in \mathbb{N}(x = y^2)$$

40. Prove $(A \subseteq B) \rightarrow (C \backslash B \subseteq C \backslash A)$.

41. Prove by deduction that $(p \wedge q) \rightarrow (p \vee q)$.

42. Show by counterexample that the following is not a tautology: $(p \rightarrow q) \rightarrow ((p \wedge r) \leftrightarrow q)$

43. Prove by deduction that $(p \vee (p \wedge q)) \leftrightarrow p$.

44. Prove by deduction that $((x \rightarrow y) \wedge (y \rightarrow z)) \rightarrow (x \rightarrow z)$

45. What is the contrapositive form of an inverse implication?