Homework 6, Math 251-010

Wednesday, February 22, 2012, due February 29, 2012

This home covers material from Sections 3.7, 3.8, 4.1, and 4.2. You should practice the problems at the ends of these sections before starting this homework.

1. Problem 7, Section 3.7. Find the general solution of
$$\ddot{y} + 4\dot{y} + 4y = t^{-2}e^{-2t}$$

2. A harmonic oscillator with mass of 11 kilograms is displaced 30 centimeters from its rest position. At this point, the spring exerts a force of 5 Newtons on the mass. The mass is then released, with no initial velocity.
   (a) If there is no friction, what will be the oscillator's period, once the mass is released?
   (b) What's the smallest possible coefficient of friction that will make the period of oscillations infinite? (In this case, the system will be critically damped.)
   (c) What's the maximum force exerted by air resistance in the case of critical damping, given the problem's current setup?

3. Simple model a car suspension system is described as follows. A 500 kg mass rests on a spring, compressing the spring 10 centimeters. On top of this mass rests another spring with a 70 kg mass on top of that that compresses it's spring 6 centimeters.

   Construct a system of equations describing the motion of the two masses in the absence of any friction. (DO NOT TRY TO SOLVE!) Notes: You will need 2 variables instead of 1. Start from first principles with explicit coordinates and Newton's second law.

4. Suppose the function $u(t) = e^{-t/10}\sin(t)$ is the solution of an under-damped harmonic oscillator.
   (a) Calculate $\dot{u}(t)$.
   (b) Plot the parametric curve $(u(t), \dot{u}(t))$ for $t \in [0, 4\pi]$. This is called a phase-plane plot, and is a very useful geometric tool for autonomous 2nd-order systems.

5. Consider the forced free harmonic oscillator
$$\ddot{v} + 9v = 4\cos \pi t$$
   with $v(0) = \dot{v}(0) = 0$.
   (a) Find the solution. Write your solution as a product of trigonometric functions. (Hints are present in section 3.8)
   (b) Find a simple but good upper bound on the amplitude of the solution.

6. If we are given the homogeneous constant-coefficient linear equation
$$\frac{dy^4}{dt^4} - 6\frac{dy^3}{dt^3} - 35\frac{dy^2}{dt^2} - 48\frac{dy}{dt} - 20y = 0,$$
   which of the following should not be in the fundamental solution set?

A.	e^{-t}	B.	te^{-t}	C.	e^{4t}
D.	e^{-2t}	E.	te^{-2t}	F.	e^{10t}

7. Find the general solution of the fourth-order homogeneous constant-coefficient equation
$$y'''' - 2y''' - 11y'' + 12y' + 36y = 0.$$
   Hint: e^{3x} is one solution. Reduction-of-order might be useful.

8. Consider the fourth-order homogeneous constant-coefficient equation
$$-360\frac{\partial}{\partial t}\,\mathrm{x}(t) + 122\frac{\partial^2}{\partial^2 t}\,\mathrm{x}(t) + 27\frac{\partial^3}{\partial^3 t}\,\mathrm{x}(t) + \frac{\partial^4}{\partial^4 t}\,\mathrm{x}(t) = 0.$$
   (a) Find the general solution.
   (b) Given initial conditions $x(0) = 4, \dot{x}(0) = -27, \ddot{x}(0) = 485, \dddot{x}(0) = -8721$, find the specific solution.

Challenge:

The Messnier equation is a linear homogeneous second-order ODE that was originally used to explain why railroad tracks in the 1800's would spontaneously break. One way to write it is

$$\ddot{x} + (1 + qu(t))x = 0$$
$$u(t) = \begin{cases} 1 & \text{if } \sin wt > 0, \\ -1 & \text{if } \sin wt < 0, \end{cases}$$

The Messnier equation is interesting because it describes a system where the ability to force the system is limited by how far the system is displaced from rest. It is also interesting because it can be exactly solved using pieces of constant-coefficient equation solutions. The Messnier equation is our simplest example of complex resonance structures.

Assume $q = 1/4$, $w = 1$, $x(0) = 0$, $x'(0) = 1$. Find $x(2\pi)$.