One-Sided Green's Functions for Common 1st and 2nd Order Linear Ordinary Differential Equation Initial Value Problems

For the first-order variable-coefficient inhomogeneous linear equation

$$\frac{dy}{dx} + q(x)y = f(x)$$

the particular solution given by integrating against the Green function is

$$y_p(x) = \int_0^x f(u)e^{-\int_u^x q(v)\mathrm{d}v}\mathrm{d}u$$

For the second-order variable-coefficient inhomogeneous linear equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x)$$

with general solution $y(x) = C_1y_1(x) + C_2y_2(x) + y_p(x)$. Then there exists a particular solution $y_p(x)$ of the form

$$y_p(x) = \int_0^x \frac{y_1(u)y_2(x) - y_1(x)y_2(u)}{y_1(u)y_2'(u) - y_1'(u)y_2(u)}f(u)du = \int_0^x g(x,u)f(u)du,$$

where the Green's function $g(x,u)$ is sometimes known from the following table.

Homogeneous ODE	Green's Function $g(x,u)$
$y' - ay$	$e^{a(x-u)}$
y''	$x-u$
$y'' - 2ay' + a^2$	$(x-u)e^{a(x-u)}$
$y'' - (a+b)y' + aby$	$\dfrac{e^{a(x-u)} - e^{b(x-u)}}{a-b}$
$y'' + b^2y$	$\dfrac{1}{b}\sin\left[b(x-u)\right]$
$y'' - b^2y$	$\dfrac{1}{b}\sinh\left[b(x-u)\right]$
$y'' - 2ay' + (a^2+b^2)y$	$\dfrac{1}{b}e^{a(x-u)}\sin\left[b(x-u)\right]$
$y'' - 2ay' + (a^2-b^2)y$	$\dfrac{1}{b}e^{a(x-u)}\sinh\left[b(x-u)\right]$
$x^2y'' + xy' - b^2y$	$\dfrac{u}{2b}\left(\dfrac{x^b}{u^b} - \dfrac{u^b}{x^b}\right)$
$x^2y'' - (b+a-1)xy' + aby$	$\dfrac{u}{b-a}\left(\dfrac{x^b}{u^b} - \dfrac{x^a}{u^a}\right)$