Homework 13, Math 250-07

not grade

This home covers material from Sections 7.,6, 7.8 and 9.1. You *must* practice the problems at the ends of these sections before starting this homework if you really want to understand things.

1. Consider each of the following first order linear systems (note, these are the same as on the previous homework)

   - Classify the geometry of behaviors of the general solutions of the following linear systems near the steady-state solution $[0, 0]$.
   - Draw a phase plane for each.

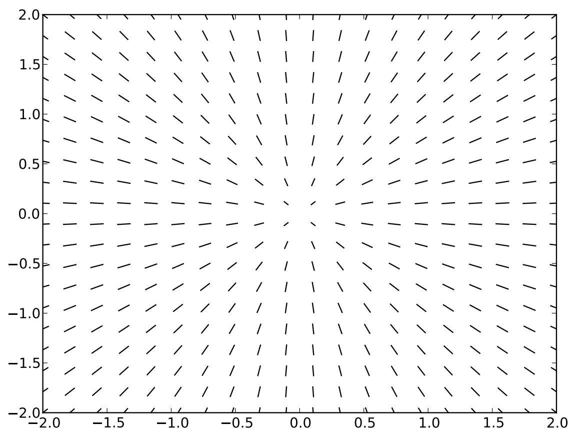
   (a)

$$\frac{dx}{dt} = Ax, \quad \text{where} \quad A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}.$$

Answer: *repeated eigenvalue -3, with any choice of linearly independent eigenvectors working. For instance,*

$$x(t) = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-3t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-3t}$$

The stationary solution $(0, 0)$ is a stable node. For the specific solution, $C_1 = 2$ and $C_2 = 1$.

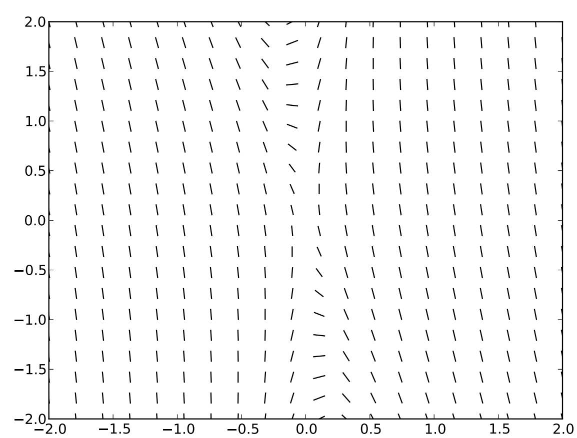


   (b)

$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} 3 & -1 \\ -24 & -2 \end{bmatrix}.$$

Answer: *The eigenvalues are -5 with eigenvector $[-1, -8]$ and 6 with eigenvector $[1, -3]$. The stationary solution $(0, 0)$ is a saddle point.*

$$y(t) = C_1 \begin{bmatrix} 1 \\ 8 \end{bmatrix} e^{-5t} + C_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{6t}$$



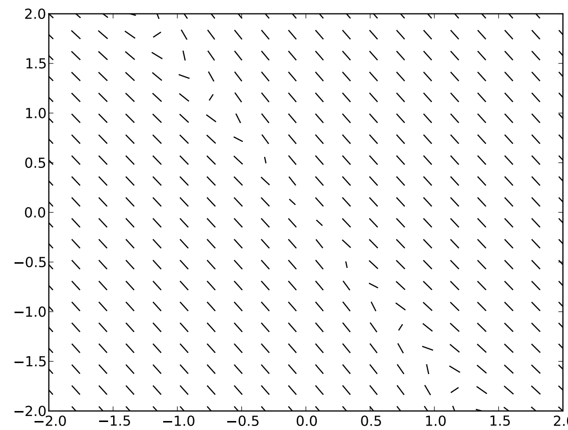
For the specific solution, $C_1 = 7/11$ and $C_2 = 15/11$.

(c)

$$\frac{dx}{dt} = Ax, \quad \text{where} \quad A = \begin{bmatrix} 13 & 8 \\ -18 & -12 \end{bmatrix}.$$

Answer: *The eigenvalues are* -3 *with eigenvector* $[-1, 2]$ *and* 4 *with eigenvector* $[8, -9]$. *The stationary solution* $(0, 0)$ *is a saddle point.*

$$x(t) = C_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-3t} + C_2 \begin{bmatrix} 8 \\ -9 \end{bmatrix} e^{4t}$$



For the specific solution, $C_1 = 26/7$ *and* $C_2 = 5/7$.

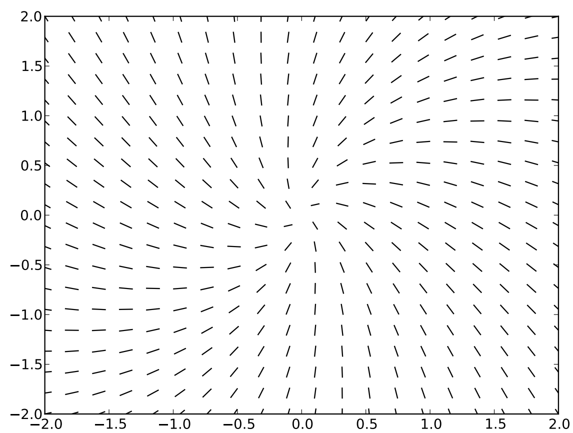
(d)

$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} -6 & -1 \\ 4 & -6 \end{bmatrix}.$$

Answer: *The eigenvalues are* $-6 \pm 2i$ *with solution*

$$y(t) = C_1 \begin{bmatrix} e^{-6t} \cos(2t) \\ 2e^{-6t} \sin(2t) \end{bmatrix} + C_2 \begin{bmatrix} e^{-6t} \sin(2t) \\ -2e^{-6t} \cos(2t) \end{bmatrix}.$$

This stationary solution is a stable focus/spiral. For the specific solution, $C_1 = 2$ *and* $C_2 = -1/2$.



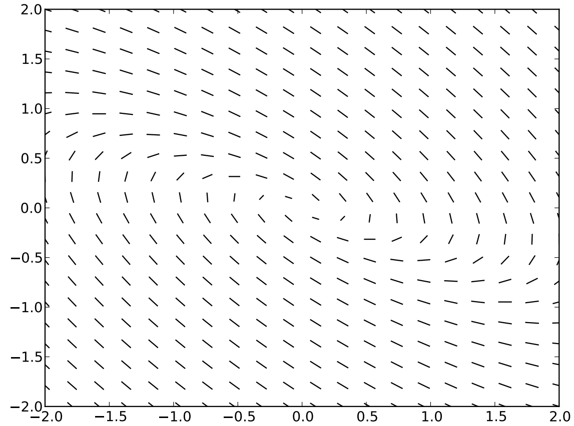
(e)

$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} -1 & -6 \\ 3 & 5 \end{bmatrix}.$$

Answer: *The eigenvalues are* $2 \pm 3i$ *with solution vectors*

$$y(t) = C_1 \begin{bmatrix} 6e^{2t} \cos(3t) \\ 6e^{2t} \sin(3t) \end{bmatrix} + C_2 \begin{bmatrix} 3e^{2t} \sin(3t) - 3e^{2t} \cos(3t) \\ -3e^{2t} \sin(3t) - 3e^{2t} \cos(3t) \end{bmatrix}$$

This stationary solution is an unstable focus/spiral. For the specific solution, $C_1 = 1/3$ *and*



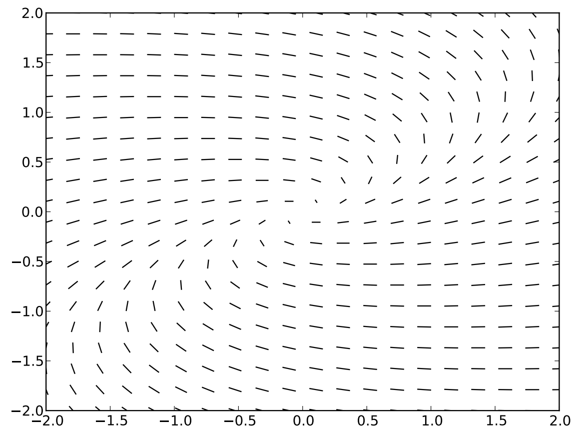
$C_2 = -2/3$.

(f)

$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}.$$

Answer:

$$y(t) = C_1 \begin{bmatrix} 4e^{2t}\cos\left(\sqrt{3}t\right) \\ \sqrt{3}e^{2t}\sin\left(\sqrt{3}t\right) + e^{2t}\cos\left(\sqrt{3}t\right) \end{bmatrix} + C_2 \begin{bmatrix} 4e^{2t}\sin\left(\sqrt{3}t\right) \\ e^{2t}\sin\left(\sqrt{3}t\right) - \sqrt{3}e^{2t}\cos\left(\sqrt{3}t\right) \end{bmatrix}.$$



The specific solution has $C_1 = \frac{1}{2}$, $C_2 = -\frac{1}{6}\sqrt{3}$.

(g)

$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}.$$

Answer: *This matrix has* $\lambda = -3$ *as a repeated eigenvalue. There is only one eigenvector,* $[1, 1]$, *for this eigenvalue. So, we look for solutions* $(\vec{\zeta} + \vec{\eta}t)e^{-3t}$. *We know from class and our textbook that* $\eta = [1, 1]$ *and* $(A + 3I)\zeta = \eta$.

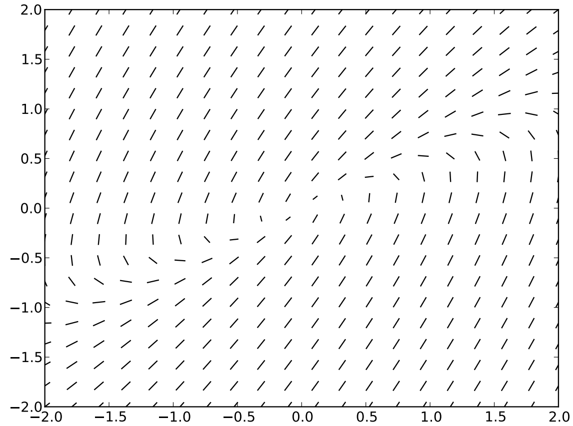
$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

There are many solutions to this equation, the simplest being $\zeta = [1, -1]$. *Our full solution then comes out as*

$$y(t) = C_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} t+1 \\ t-1 \end{bmatrix}$$

This is a stable improper node. The specific solution

$$y(t) = e^{-3t} \begin{bmatrix} t/2 + 2 \\ t/2 + 1 \end{bmatrix}$$



(h)

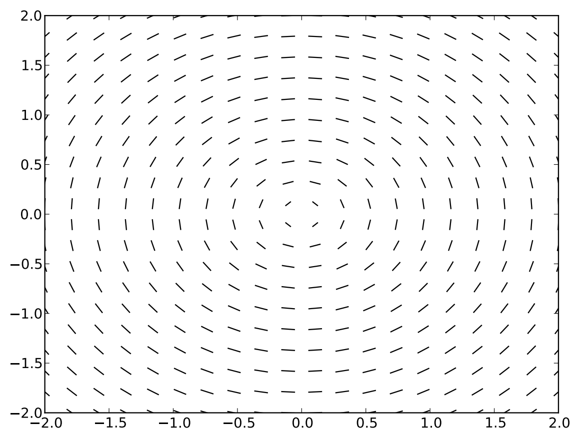
$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}.$$

Answer: *center, imaginary eigenvalues* $\pm 4i$.

$$y(t) = C_1 \begin{bmatrix} \sin(4t) \\ \cos(4t) \end{bmatrix} + C_2 \begin{bmatrix} \cos(4t) \\ -\sin(4t) \end{bmatrix}$$

The specific solution

$$y(t) = \begin{bmatrix} \sin(4t) + 2\cos(4t) \\ \cos(4t) - 2\sin(4t) \end{bmatrix}$$



2. Consider the following first order linear systems with repeated roots.

   - Find the general solution of this system.
   - Classify the behavior around the stationary solution and draw a phase portrait.

(a)

$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} 4 & -2 \\ 2 & 0 \end{bmatrix}.$$

Answer: *There is a repeated eigenvalue* 2 *leading to general solution*

$$x(t) = e^{2t} \left[C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} \right) \right]$$

(note, there is more than one way to write this solution that is correct) The stationary point $(0, 0)$ *is unstable.*

(b)

$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} 0 & -2 \\ 8 & -8 \end{bmatrix}.$$

Answer: *There is a repeated eigenvalue* -4*, with one eigenvector* $v = [1, 2]$*. But this is the only one, so we need to look for a generalized eigenvector to construct our solutions. We would like to find* $x(t) = (u + wt)e^{\lambda t}$ *where* $(A - \lambda I)w = 0$ *and* $(A - \lambda I)u = w$*. We find* $w = k[1, 2]$*. From this, the second equation gives* $4u_1 - 2u_2 = k$*.*

$$x(t) = e^{-4t}\left[\begin{bmatrix}\frac{k+2u_2}{4}\\ u_2\end{bmatrix} + kt\begin{bmatrix}1\\2\end{bmatrix}\right]$$
$$x(t) = e^{-4t}\left(C_1\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix} + C_2\begin{bmatrix}t+\frac{1}{4}\\2t\end{bmatrix}\right)$$

If we use an orthogonal basis, we can rewrite this as

$$x(t) = e^{-4t}\left[C_1\begin{bmatrix}4\\8\end{bmatrix} + C_2\left(t\begin{bmatrix}4\\8\end{bmatrix} + \begin{bmatrix}\frac{4}{5}\\-\frac{2}{5}\end{bmatrix}\right)\right]$$

(note, there is more than one way to write this solution that is correct) To check this solution, take $C_1 = 0$*,* $C_2 = 1$*. Then*

$$x' = \begin{bmatrix}-4\left(4t+\frac{4}{5}\right)e^{-4t}+4e^{-4t}\\ -4\left(8t-\frac{2}{5}\right)e^{-4t}+8e^{-4t}\end{bmatrix}$$
$$Ax = \begin{bmatrix}-2\left(8t-\frac{2}{5}\right)e^{-4t}\\ 8\left(4t+\frac{4}{5}\right)e^{-4t}-8\left(8t-\frac{2}{5}\right)e^{-4t}\end{bmatrix} x' = Ax.$$

3. Linear Inhomogeneous systems can be written $\dot{x} = Ax + f(t)$. Consider the linear inhomogeneous system

$$\begin{bmatrix}\frac{dx_1}{dt}\\ \frac{dx_2}{dt}\end{bmatrix} = \begin{bmatrix}4 & 3\\ 4 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}e^t\\e^t\end{bmatrix}.$$

   (a) Find the homogeneous solution.
   (b) Find the inverse of the eigenvector matrix T, and and use it to transform the system into decoupled equations for $y_p(t)$.
   (c) Find particular solutions for $y_p(t)$.
   (d) Transform your particular solutions for $y_p(t)$ back to particular solutions $x_p(t)$ using $x_p(t) = Ty_p(t)$.
   (e) Construct the general solution for $x(t)$ by summing the particular and homogeneous solutions.

4. Linear Inhomogeneous systems can be written $\dot{x} = Ax + f(t)$. Consider the linear inhomogeneous system

$$\begin{bmatrix}\frac{dx_1}{dt}\\ \frac{dx_2}{dt}\end{bmatrix} = \begin{bmatrix}2 & -7\\ -5 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}3\\t\end{bmatrix}.$$

   (a) Find the homogeneous solution.
   (b) Find the inverse of the eigenvector matrix T, and and use it to transform the system into decoupled equations for $y_p(t)$.
   (c) Find particular solutions for $y_p(t)$.
   (d) Transform your particular solutions for $y_p(t)$ back to particular solutions $x_p(t)$ using $x_p(t) = Ty_p(t)$.
   (e) Construct the general solution for $x(t)$ by summing the particular and homogeneous solutions.