Homework 12, Math 250-07

due Wednesday, December 5, 2012

This home covers material from Sections 7.1, 7.2, and 7.3, 7.5, 7.6, 7.8. You *must* practice the problems at the ends of these sections before starting this homework if you really want to understand things.

1. What do theorems 7.1.1 and 7.1.2 say?

   Answer: *Theorem 7.1.1 provides conditions for when a nonlinear system can be solved and has a unique solution. The basic condition is that if $\dot{x} = F(x,t)$ then there is a solution near the initial condition as long as F and $\partial F/\partial x$ are locally continuous in x. Theorem 7.1.2 extends this result to linear systems, where we can relax the conditions to only requiring continuity of the linear system's coefficients, and the solution can be extended to the whole interval of continuity.*

2. Section 7.2, 371-373, # 1 Answer:

   (a) # 1

   $$A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -2 & 3 \\ -1 & 5 & 0 \\ 6 & 1 & 2 \end{bmatrix},$$

   i.

   $$2A + B = \begin{bmatrix} 6 & -6 & 3 \\ 5 & 9 & -2 \\ 2 & 3 & 8 \end{bmatrix}$$

   ii.

   $$A - 4B = \begin{bmatrix} -15 & 6 & -12 \\ 7 & -18 & -1 \\ -26 & -3 & -5 \end{bmatrix}$$

   iii.

   $$AB = \begin{bmatrix} 6 & -12 & 3 \\ 4 & 3 & 7 \\ 9 & 12 & 0 \end{bmatrix}$$

   iv.

   $$BA = \begin{bmatrix} -8 & -9 & 11 \\ 14 & 12 & -5 \\ 5 & -8 & 5 \end{bmatrix}$$

3. Section 7.2, 371-373, # 22. Answer: *Show that if $x = (4,2)e^{2t}$, then*

   $$\dot{x} = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} x$$

   by substitution. No special answer.

4. Section 7.2, 371-373, # 24 Answer: *Another check problem. No special answer.*

5. Give an example of a 4x4 symmetric matrix that is not diagonal. What do we know about the eigenvalues of this matrix? Answer: *The eigenvalues of a symmetric matrix are always real, never complex or imaginary*

6. Find the eigenvalues and eigenvectors of the following matrices.

   (a)

   $$\begin{bmatrix} -16 & -54 \\ 4 & 14 \end{bmatrix}$$

   Answer: *The eigenvalues are -4 with eigenvector $[9, -2]$ and 2 with eigenvector $[6, -2]$.*

   (b)

   $$\begin{bmatrix} 70 & -60 \\ 72 & -62 \end{bmatrix}$$

   Answer: *The eigenvalues are 10 with eigenvector $[-8, -8]$ and -2 with eigenvector $[5, 6]$.*

(c)

$$\begin{bmatrix} 2 & 5 \\ -5 & 2 \end{bmatrix}$$

Answer: *The eigenvalues are* $2 \pm 5i$, *with eigenvectors* $(-i, 1)$ *and* $(i, 1)$.

7. Consider each of the following first order linear systems

- Find the general solution of this system.
- Find the specific solution if $y(0) = [2, 1]$.

(a)

$$\frac{dx}{dt} = Ax, \quad \text{where} \quad A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}.$$

Answer: *repeated eigenvalue* -3, *with any choice of linearly independent eigenvectors working. For instance,*

$$x(t) = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-3t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-3t}$$

The stationary solution $(0, 0)$ *is a stable node. For the specific solution,* $C_1 = 2$ *and* $C_2 = 1$.

(b)

$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} 3 & -1 \\ -24 & -2 \end{bmatrix}.$$

Answer: *The eigenvalues are* -5 *with eigenvector* $[-1, -8]$ *and* 6 *with eigenvector* $[1, -3]$. *The stationary solution* $(0, 0)$ *is a saddle point.*

$$y(t) = C_1 \begin{bmatrix} 1 \\ 8 \end{bmatrix} e^{-5t} + C_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{6t}$$

For the specific solution, $C_1 = 7/11$ *and* $C_2 = 15/11$.

(c)

$$\frac{dx}{dt} = Ax, \quad \text{where} \quad A = \begin{bmatrix} 13 & 8 \\ -18 & -12 \end{bmatrix}.$$

Answer: *The eigenvalues are* -3 *with eigenvector* $[-1, 2]$ *and* 4 *with eigenvector* $[8, -9]$. *The stationary solution* $(0, 0)$ *is a saddle point.*

$$x(t) = C_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-3t} + C_2 \begin{bmatrix} 8 \\ -9 \end{bmatrix} e^{4t}$$

For the specific solution, $C_1 = 26/7$ *and* $C_2 = 5/7$.

(d)

$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} -6 & -1 \\ 4 & -6 \end{bmatrix}.$$

Answer: *The eigenvalues are* $-6 \pm 2i$ *with solution*

$$y(t) = C_1 \begin{bmatrix} e^{-6t} \cos(2t) \\ 2e^{-6t} \sin(2t) \end{bmatrix} + C_2 \begin{bmatrix} e^{-6t} \sin(2t) \\ -2e^{-6t} \cos(2t) \end{bmatrix}.$$

This stationary solution is a stable focus/spiral. For the specific solution, $C_1 = 2$ *and* $C_2 = -1/2$.

(e)

$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} -1 & -6 \\ 3 & 5 \end{bmatrix}.$$

Answer: *The eigenvalues are* $2 \pm 3i$ *with solution vectors*

$$y(t) = C_1 \begin{bmatrix} 6e^{2t} \cos(3t) \\ 6e^{2t} \sin(3t) \end{bmatrix} + C_2 \begin{bmatrix} 3e^{2t} \sin(3t) - 3e^{2t} \cos(3t) \\ -3e^{2t} \sin(3t) - 3e^{2t} \cos(3t) \end{bmatrix}$$

This stationary solution is an unstable focus/spiral. For the specific solution, $C_1 = 1/3$ *and* $C_2 = -2/3$.

(f)

$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}.$$

Answer:

$$y(t) = C_1 \begin{bmatrix} 4e^{2t}\cos\left(\sqrt{3}t\right) \\ \sqrt{3}e^{2t}\sin\left(\sqrt{3}t\right) + e^{2t}\cos\left(\sqrt{3}t\right) \end{bmatrix} + C_2 \begin{bmatrix} 4e^{2t}\sin\left(\sqrt{3}t\right) \\ e^{2t}\sin\left(\sqrt{3}t\right) - \sqrt{3}e^{2t}\cos\left(\sqrt{3}t\right) \end{bmatrix}.$$

The specific solution has $C_1 = \frac{1}{2}$, $C_2 = -\frac{1}{6}\sqrt{3}$.

(g)

$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}.$$

Answer: *This matrix has* $\lambda = -3$ *as a repeated eigenvalue. There is only one eigenvector,* $[1, 1]$, *for this eigenvalue. So, we look for solutions* $(\vec{\zeta} + \vec{\eta}t)e^{-3t}$. *We know from class and our textbook that* $\eta = [1, 1]$ *and* $(A + 3I)\zeta = \eta$.

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

There are many solutions to this equation, the simplest being $\zeta = [1, -1]$. *Our full solution then comes out as*

$$y(t) = C_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} t+1 \\ t-1 \end{bmatrix}$$

This is a stable improper node. The specific solution

$$y(t) = e^{-3t} \begin{bmatrix} t/2 + 2 \\ t/2 + 1 \end{bmatrix}$$

(h)

$$\frac{dy}{dt} = Ay, \quad \text{where} \quad A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}.$$

Answer: *center, imaginary eigenvalues* $\pm 4i$.

$$y(t) = C_1 \begin{bmatrix} \sin(4t) \\ \cos(4t) \end{bmatrix} + C_2 \begin{bmatrix} \cos(4t) \\ -\sin(4t) \end{bmatrix}$$

The specific solution

$$y(t) = \begin{bmatrix} \sin(4t) + 2\cos(4t) \\ \cos(4t) - 2\sin(4t) \end{bmatrix}$$

8. Transform the following equations into systems of 1st order ODE's.

(a)

$$\ddot{y} - 5\dot{y} + y = 0.$$

Answer: *Let* $x_0(t) := y(t)$ *and* $x_1(t) := \dot{y}(t)$ *so*

$$\dot{x}_0 = x_1$$
$$\dot{x}_1 = 5x_1 - x_0$$

(b)

$$\ddot{y} - y\dot{y} + y^3 = t.$$

Answer: *Let* $x_0(t) := y(t)$ *and* $x_1(t) := \dot{y}(t)$ *so*

$$\dot{x}_0 = x_1$$
$$\dot{x}_1 = x_0 x_1 - x_0^3 + t$$

(c)

$$t^2\dddot{y} - \dot{y} + ty = 3, \; y(0) = \dot{y}(0) = 1.$$

Answer: *Let* $x_0(t) := y(t)$, $x_1(t) := \dot{y}(t)$, *and* $x_2(t) := \ddot{y}(t)$ *so*

$$\begin{aligned} \dot{x}_0 &= x_1 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{x_1 - tx_0 + 3}{t^2} \end{aligned}$$

with initial conditions $x_0(0) = 1$ *and* $x_1(0) = 1$. *(Note that these conditions are not enough to identify a unique solution, but since you aren't asked to solve things, that's not an issue.)*