Homework 10, Math 251-007

Sunday, November 11, 2012, due Wednesday, November 14, 2012

This home covers material from Sections 6.3, and 6.4. You should practice the problems at the ends of these sections before starting this homework.

1. Represent the following functions in terms of polynomials and Heaviside functions.

   (a)
   $$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2 \\ -1 & \text{if } 2 \le x < 3 \\ -4 & \text{if } 3 \le x \end{cases}$$

   Answer: $f(x) = u_1(x) - 2u_2(x) - 3u_3(x)$

   (b) $g(x) = |x-2| + |x+2|$

   Answer: $g(x) = -2x + (2x+4)u_{-2}(x) + (2x-4)u_2(x)$

2. Solve the following initial-value problems using Laplace transforms.

   (a)
   $$\dot{y} - 3y = u_2(t) + u_{10}(t), y(0) = 0$$

   Answer:
   $$(s-3)\mathcal{L}y = y(0) + e^{-2s}/s + e^{-10s}/s$$
   $$\mathcal{L}y = \frac{e^{-2s}}{s(s-3)} + \frac{e^{-10s}}{s(s-3)}$$
   $$y(t) = u_2(t)\frac{e^{3(t-2)} - 1}{3} + u_{10}(t)\frac{e^{3(t-10)} - 1}{3}$$

   (b)
   $$\ddot{y} + 4y = u_2(t) - u_3(t), y(0) = \dot{y}(0) = 0$$

   Answer:
   $$\mathcal{L}y = \frac{e^{-2s}}{s(s^2+4)} - \frac{e^{-3t}}{s(s^2+4)}$$
   $$y(t) = \frac{1}{4}\left[(1-\cos(2(t-2)))u_2(t) - (1-\cos(2(t-3)))u_3(t)\right]$$

   (c)
   $$\ddot{y} + 3\dot{y} + 2y = 1 - u_9(t), y(0) = \dot{y}(0) = 0$$

   Answer:
   $$\mathcal{L}y = \frac{1 - e^{-9s}}{s\,(s^2+3s+2)}$$
   $$\mathcal{L}y = (1 - e^{-9s})\left[\frac{1}{2(s+2)} - \frac{1}{s+1} + \frac{1}{2s}\right]$$
   $$y(t) = \left[\frac{1}{2}e^{-2t} - e^{-t} + \frac{1}{2}\right] - u_9(t)\left[\frac{1}{2}e^{-2(t-9)} - e^{-(t-9)} + \frac{1}{2}\right]$$

3. Find the specific solution to the following initial-value problems using Laplace--Heaviside methods.

   (a)
   $$y'' + 4y' + 3y = \delta(x-12), \quad y(0) = 0, \quad y'(0) = 0.$$

   Answer:
   $$y(x) = u_{12}(x)\frac{1}{2}\left(e^{12-x} - e^{36-3x}\right)$$

(b)

$$\ddot{y} + y = \delta(t+2) + u_3(t), \quad y(0) = 0, \quad \dot{y}(0) = 0.$$

Answer: *The trick here is that the Dirac delta-function does not spike in the Laplace-transform interval (it happens before our initial conditions start) and disappears when we do our integral. So really,*

$$s^2\mathcal{L}y - sy(0) - \dot{y}(0) + \mathcal{L}y = e^{-3s}/s$$
$$\mathcal{L}y = \frac{e^{-3s}}{s(s^2+1)}$$
$$y(t) = \mathcal{L}^{-1}\left[e^{-3s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\right]$$
$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] = 1 - \cos t$$
$$y(t) = u_3(t)\left[1 - \cos(t-3)\right]$$

(c)

$$\ddot{y} + 20\dot{y} + 109y = \delta(t-1), \quad y(0) = 2, \quad \dot{y}(0) = 0.$$

Answer:

$$\mathcal{L}y = \frac{(2s+40) + e^{-s}}{s^2 + 20s + 109}$$
$$\mathcal{L}y = \frac{2(s+10) + 20 + e^{-s}}{(s+10)^2 + 9}$$
$$y(t) = \frac{20}{3}e^{-10t}\sin(3t) + 2e^{-10t}\cos(3t) + \frac{1}{3}u_1(t)e^{-10(t-1)}\sin(3(t-1))$$