Homework 9, Math 251-007

Friday, November 2, 2012, due Wednesday, November 7, 2012

This home covers material from Sections 6.2. You should practice the problems at the ends of these sections before starting this homework.

1. Solve the following initial-value problems using Laplace-transform methods.

(a)

$$\ddot{c} - 4\dot{c} - 21c = 0, \quad c(0) = 4, \quad \dot{c}(0) = 8.$$

Answer:

$$c(t) = \mathcal{L}^{-1}\left\{\frac{sc(0) + \dot{c}(0) - 4c(0)}{s^2 - 4s - 21}\right\} = \mathcal{L}^{-1}\left\{\frac{4s-8}{(s-7)(s+3)}\right\} = 2(e^{-3t} + e^{7t}).$$

(b)

$$\ddot{c} - 4\dot{c} - 21c = t, \quad c(0) = 4, \quad \dot{c}(0) = 8.$$

Answer:

$$c(t) = \mathcal{L}^{-1}\left\{\frac{4s-8}{(s-7)(s+3)} + \frac{1}{s^2(s-7)(s+3)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{4s-8}{(s-7)(s+3)} - \frac{1}{90(s+3)} + \frac{1}{490(s-7)} + \frac{4}{441s} - \frac{1}{21s^2}\right\}$$

$$= 2(e^{-3t} + e^{7t}) - \frac{e^{-3t}}{90} + \frac{e^{7t}}{490} + \frac{4}{490} - \frac{t}{21}$$

This can be simplified a little if desired.

(c)

$$\ddot{c} - 4\dot{c} - 21c = e^{-t}, \quad c(0) = 4, \quad \dot{c}(0) = 8.$$

Answer:

$$c(t) = \mathcal{L}^{-1}\left\{\frac{4s-8}{(s-7)(s+3)} + \frac{1}{(s+1)(s-7)(s+3)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{4s-8}{(s-7)(s+3)} + \frac{1}{20(s+3)} - \frac{1}{16(s+1)} + \frac{1}{80(s-7)}\right\}$$

$$= 2(e^{-3t} + e^{7t}) + \frac{e^{-3t}}{20} - \frac{e^{-t}}{16} + \frac{e^{7t}}{80}$$

This can be simplified a little if desired.

(d)

$$\ddot{y} - 2\dot{y} + 29y = e^{-t}, \quad y(0) = 1, \quad \dot{y}(0) = 0.$$

Answer: *This equation leads to complex eigenvalues. Taking the Laplace transform of both sides leads to*

$$\mathcal{L}\{y\} = \frac{(sy(0) + \dot{y}(0) - 2y(0))}{s^2 - 2s + 29} + \frac{1}{(s+1)(s^2 - 2s + 29)}$$

$$= \frac{(s-2)}{(s-1)^2 + 28} + \frac{1}{(s+1)((s-1)^2 + 28)}$$

$$= \frac{(s-2)}{(s-1)^2 + 28} - \frac{s-3}{32((s-1)^2 + 28)} + \frac{1}{32(s+1)}$$

$$= \frac{\frac{31}{32}s - \frac{61}{32}}{(s-1)^2 + 28} + \frac{1}{32(s+1)}$$

$$y(t) = \frac{31}{32}e^t \cos(2\sqrt{7}t) - \frac{15}{32\sqrt{7}}e^t \sin(2\sqrt{7}t) + \frac{e^{-t}}{32}$$

(e)

$$\ddddot{x} - x = 0, \quad \ddot{x}(0) = 1, \quad x(0) = \dot{x}(0) = \dddot{x}(0) = 0.$$

Answer: *Transforming the equation,*

$$s^4\mathcal{L}\{x\} - s^3x(0) - s^2\dot{x}(0) - s\ddot{x}(0) - \dddot{x}(0) - \mathcal{L}\{x\} = 0$$

So

$$\mathcal{L}\{x\} = \frac{s}{s^4 - 1}.$$

$$\mathcal{L}\{x\} = \frac{1}{4(s+1)} + \frac{1}{4(s-1)} - \frac{1}{2}\frac{s}{s^2+1}$$

$$x(t) = e^{-t}/4 + e^t/4 - \frac{1}{2}\cos t$$

or

$$x(t) = \frac{1}{2}(\cosh(t) - \cos t)$$